Tertulia Seminar.

Descriptive Family of Banach Spaces.

Nulla è ancora, e qualcosa deve divenire. Il cominciamento non è il puro nulla, ma un nulla da cui deve uscire qualcosa. Dunque anche nel cominciamento è già contenuto l'essere. Il cominciamento contien dunque l'uno e l'altro, l'essere e il nulla; è l'unità dell'essere col nulla; – ossia è un non essere, che è in pari tempo essere, e un essere, che è in pari tempo un non essere.
Oltracciò l'essere e il nulla son nel cominciamento come diversi; poichè il cominciamento accenna a qualcos'altro; – è un non essere che si riferisce all'essere come a un altro; ciò che comincia non *è* ancora; va, soltanto, all'essere. Il cominciamento contien dunque l'essere come quello che si allontana dal non essere, o lo toglie via considerandolo come contrapposto a lui.
Ma, inoltre, quello che comincia *è* già; in pari tempo però non è ancora. Nel cominciamento dunque, questi opposti, l'essere e il non essere, sono immediatamente uniti. Vale a dire che il cominciamento è la loro unità indifferente, indistinta.

G.W.F. Hegel
Scienza della Logica

Chapter 1

Some Descriptive Set Theory

We review first some of the basic concepts and results of descriptive set theory in Polish spaces. References for the proofs of the theorems that we do not give are Kuratowski [9, Vol. I, §33-39] and Kechris [7].

Throughout these notes, we denote by $\omega = \{0, 1, \ldots, \}$ the first infinite ordinal, let $\omega^{<\omega}$ be the set of all finite sequence in ω and let us denote by ω^ω be the set of all sequence of elements of ω. If $s = (s(0), ..., s(n-1))$ is an element of $\omega^{<\omega}$, we denote its length n by $|s|$. In particular the empty sequence $\emptyset$ has length 0. For $s = (s(0), ..., s(n-1)), t = (t(0), ..., t(k-1))$ the concatenation $s \frown t$ is defined by

$$s \frown t = (s(0), ..., s(n-1), t(0), ..., t(k-1)).$$

For $x \in \omega^\omega$, and $n \in \omega$ in the sequel, we will consider the following notation

$$x|n = (x(0), \ldots, x(n-1))$$

Definition 1.1. A topological space is *completely metrizable* if it admits a complatible metric d such that (X, d) is complete. A separable completely metrizable space is called *Polish*.

Proposition 1.2. *The following facts hold*

i) *The completion of a separable metric space is Polish;*

ii) *A closed subspace of a Polish space is Polish;*

iii) *The product of a sequence of completely metrizable (resp. Polish) spaces is completely metrizable (resp. Polish);*

iv) *A subspace of a Polish space is Polish if and only if it is a G_δ (intersection of countable many open sets).*

Proof. $(i), (ii)$ and (iii) are easy. Let us sketch the proof of (iv).

Let X be a Polish space, and let $Y = \bigcap_n U_n$ with U_n open in X. Let d be a complete compatible metric for X and consider $F_n = X \setminus U_n$ for all $n \in \omega$. Define a new metric on Y

$$d'(x, y) = d(x, y) + \sum_{n=0}^{\infty} \min \left\{ \frac{1}{2^{n+1}}, \left| \frac{1}{d(x, F_n)} - \frac{1}{d(y, F_n)} \right| \right\}$$

It is easy to check that this is a metric compatible with the topology of Y. We show that (Y, d') is complete.

Let $(y_i)_i$ be a Cauchy sequence in (Y, d'). Then it is Cauchy in (X, d). Since (X, d) is complete, there exists $y \in X$ such that $y_i \xrightarrow{d} y$. But also for each $n \in \omega$,

$$\lim_{i,j\to\infty} \left| \frac{1}{d(y_i, F_n)} - \frac{1}{d(y_j, F_n)} \right| = 0.$$

So, for each $n \in \omega$, the sequence $(\frac{1}{d(y_i,F_n)})_i$ converges in $\mathbb{R}$.Therefore, $\frac{1}{d(y_i,F_n)}$ is bounded away from 0. Since $\frac{1}{d(y_i,F_n)} \to \frac{1}{d(y,F_n)}$, we have that $d(y, F_n) \neq 0$ for all $n \in \omega$. Thus, $y \notin F_n$, for all $n \in \omega$; i.e. $y \in Y$ and clearly $y_i \xrightarrow{d'} y$ □

In particular from *iv)* every open set in a Polish space is Polish. For example, this last proposition tell us that $\{0,\ 1\}^\omega = 2^\omega$ is a Polish space (usually called as *Cantor space*).

If we consider ω with the discrete topology, from *iii)* we can say that ω^ω is Polish too. This last space (called *Baire space*) plays an important rule in the Polish spaces theory because of

Proposition 1.3. *If X is a nonempty Polish space, then there exist a closed set $\mathcal{F} \subseteq \omega^\omega$ and a continuous and bijection $f : \mathcal{F} \longrightarrow X$. In particular, if X is nonempty, there is a continuous surjection $\widetilde{f} : \omega^\omega \longrightarrow X$ extending f.*

Proof. The last assertion is clear. Let us prove the first one.

Since X are Polish spaces, in particular $X = \bigcup_{n_1 \in \omega} C(n_1)$ (where with $C(n_1)$ we are denoting the ball of centers in X and radius 1). Since $C(n_1)$ is a Polish space (because open), for the same sake, $C(n_1) = \bigcup_{n_2 \in \omega} C(n_1, n_2)$, where with $C(n_1, n_2)$ we are denoting the ball of centers in $C(n_1)$ and radius $\frac{1}{2}$. By induction we can define $C(n_1, ..., n_k)$ so that

(a) $C(n_1, ..., n_{k-1}) = \bigcup_{n_k \in \omega} C(n_1, ..., n_k) = \bigcup_{n_k \in \omega} \overline{C(n_1, ..., n_k)}$,

(b) $diam(C(n_1, ..., n_k)) < \frac{1}{k}$.

Let

$$\mathcal{D} = \{\mathbf{n} \in \omega^\omega \; : \; \bigcap_{k\in\omega} C(\mathbf{n}|k) \neq 0\}$$

Then we can define $f : \mathcal{D} \longrightarrow X$ by

$$f(\mathbf{n}) = \bigcap_{k\in\omega} C(\mathbf{n}|k)$$

where we are using $\mathbf{n} = (n_1, ..., n_m, ...)$. Let us prove first the continuity.

Let $\epsilon > 0$ arbitrary; by construction, we can suppose that the open in X is $C(n_1, ..., n_{k_1})$. Fix $k = \max\{k_1, \frac{1}{\epsilon}\}$. Let $\mathbf{n}$ so that $f(\mathbf{n}) \in C(n_1, ..., n_k)$ and we can consider the open of $\mathbf{n}$ in a such way

$$\mathcal{N}(n_1, ..., n_k) = \{\mathbf{m} \in \omega^\omega : (m_1, ..., m_k) = (n_1, ..., n_k)\}.$$

Then, we have $f(\mathbf{m}) \in C(n_1, ..., n_k)$ for each $\mathbf{m} \in \mathcal{N}(n_1, ..., n_k)$, so

$$d(f(\mathbf{n}), f(\mathbf{m})) \leq diam C(n_1, ..., n_k) < \frac{1}{k} \leq \epsilon \qquad \forall \mathbf{m} \in \mathcal{N}(n_1, ..., n_k)$$

That imply that f is continuous.

It is straightforward to check that f is injective and from (a) above $f(\mathcal{D}) = X$.

To finish, we need to show that $\mathcal{D}$ is a closed set in ω^ω.

Let us suppose $(\mathbf{x}_n)_n$ be a sequence in $\mathcal{D}$ such that $\mathbf{x}_n \overset{n\to\infty}{\longrightarrow} \mathbf{x} \in \omega^\omega$. Given $\varepsilon > 0$ there is N with $diam(C(\mathbf{x}|N)) < \varepsilon$ and M such that $\mathbf{x}_n|N = \mathbf{x}|N$ for all $n \geq M$, so that $d(f(\mathbf{x}_n), f(\mathbf{x}_m)) < \varepsilon$ if $m, n \geq M$. Therefore, $(f(\mathbf{x}_n))_n$ is Cauchy, so that $f(\mathbf{x}_n) \longrightarrow y \in X$. Then $f(\mathbf{x}) = y$. By definition of f we get $y \in \bigcap_n \overline{C(\mathbf{x}|n)} = \bigcap_n C(\mathbf{x}|n)$, so that $\mathbf{x} \in \mathcal{D}$. □

For the Cantor space, we have

Theorem 1.4. *Every nonempty compact metrizable space is a continuous image of* 2^ω.

Proof. Let $\mathbb{I} = [0,\ 1]$, and let $f : 2^\omega \longrightarrow \mathbb{I}$ the map

$$f(x) = \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}.$$

Then f maps 2^ω continuously onto $\mathbb{I}$. So

$$(x_n)_n \longmapsto (f(x_n))_n$$

maps 2^{ω^ω}, which is homeomorphic to 2^ω, onto $\mathbb{I}^\omega$.

We claim that every compact metrizable space is homeomorphic to a compact subset of $\mathbb{I}^\omega$.

Indeed, if (X, d) is a separable metric space with $d \leq 1$ and $(x_n)_n$ a dense sequence of X, then

$$f : X \longrightarrow \mathbb{I}^\omega$$

defined by

$$f(x) = (d(x, x_n))_n$$

is clearly continuous and injective. To see that $f^{-1} : f(X) \longrightarrow X$ is also continuous, let $f(x^m) \to f(x)$. That means $d(x^m, x_n) \to d(x, x_n)$ for all $n \in \omega$. Fix $\varepsilon < 0$ and consider n such that $d(x, x_n) < \varepsilon$. Since $d(x^m, x_n) \to d(x, x_n)$ we can take M such that if $m \geq M$ then $d(x^m, x_n) < \varepsilon$. Then if $m \geq M$, $d(x^m, x) < 2\varepsilon$. Thus $x^m \to x$. This prove the claim.

From the last assertion, we have that there exists a closed set $F \subseteq 2^\omega$ and a continuous surjection of F onto X. But it is well known (see [7, Proposition 2.8]) that for every closed F of 2^ω there always exists a continuous surjection $f : 2^\omega \longrightarrow F$, such that $f(x) = x$ for $x \in F$. □

Let us recall some elementary notion.

A *limit point* of a topological space is a point that is not isolated, i.e., for every open neighborhood U of x there exists a point $y \in U$ with $y \neq x$. A space is *perfect* if all of its points are limit points. If P is a subset of a topological space X, we call P perfect in X if P is closed and perfect in its relative topology. A point x is called *condensation point* if every open neighborhood of x is uncountable.

Theorem 1.5. *Let X be a nonempty perfect Polish space. Then there is an embedding of 2^ω into X.*

Proof. For each $s \in 2^{<\omega}$, we define a nonempty open subset U_s of X by induction of $|s|$.

Consider $U_\emptyset$ an arbitrary nonempty open set of X. Given U_s, choosing $x \neq y$ in U_s (since X is perfect), we define $U_{s^\frown 0}$, $U_{s^\frown 1}$ small enough balls around x, y respectively, such that

(i) U_s is open nonempty;

(ii) $diam(U_s) \leq 2^{-|s|}$;

(iii) $\overline{U_{s^\frown i}} \subseteq U_s$ for $s \in 2^\omega$, $i \in \{0, 1\}$.

Then for each $x \in 2^\omega$, $\bigcap_n U_{x|n} = \bigcap_n \overline{U_{x|n}}$ is a singleton (by completeness of X). Let define

$$f : 2^\omega \longrightarrow X$$

by

$$x \longmapsto \bigcap_n U_{x|n}.$$

Then f is injective and continuous, therefore an embedding. □

Theorem 1.6. *(Cantor-Bendixson)*
Let X be a Polish space. Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C countable open.

Proof. Let

$$X^* = \{x \in X \ : \ x \ \text{ is a condensation point of } X\}$$

Let $P = X^*$ and $C = X \setminus P$. If $\{U_n\}_n$ is an open basis of X, then C is the union of all U_n's which are countable, so C is open countable. Of course, P is closed. Let $x \in P$ and U be a open neighborhood of x. Then U is uncountable, so it contains uncountable many condensation points, and $U \cap P$ is thus uncountable.

To prove the uniqueness, Let us suppose $X = P_1 \cup C_1$ be another such decomposition. Since P_1 is perfect, then $P_1^* = P_1$ (this is because, if $x \in P_1$ and U is a neighborhood of x, then $U \cap P_1$ is perfect nonempty Polish, thus of cardinality $2^{\aleph_0}$) and thus $P_1 \subseteq P$.

Now, since C_1 is a countable open, then $C_1 \subseteq C$. Therefore $P = P_1$ and $C = C_1$. □

Corollary 1.7. *Any uncountable Polish space contains a homeomorphic copy of 2^ω.*

Remark 1.8. In the sequel, we will see that ω^ω can be identify with a G_δ subset of 2^ω (see the proof of Theorem 1.21). Therefore, any uncountable Polish space contains a homeomorphic copy of ω^ω.

Definition 1.9. Let (X, Θ) be a topological space. The class of *Borel set* of X is the σ-algebra generated by the open sets. We will denote by $\mathcal{B}(X)$ such σ-algebra.
Let X, Y be topological spaces. A map $f : X \longrightarrow Y$ is *Borel* if the inverse image of a Borel (equivalently open or closed) set is Borel.

For a Polish space X, in the sequel we will consider the following hierarchy

$$\Sigma_1^0 = \text{all open sets}$$

$$\Pi_1^0 = \{X \setminus A \;\; A \in \Sigma_1^0\} = \text{all closed sets}$$

For each countable ordinal $\alpha < \omega_1$,

$$\Sigma_\alpha^0 = \{\cup_n A_n \;:\; \text{each } A_n \text{ is in } \Pi_{\alpha_n}^0, \text{ for some } \alpha_n < \alpha\}$$

and

$$\Pi_\alpha^0 = \{X \setminus A \;\; A \in \Sigma_\alpha^0\}$$

In particular, we have

$$\Sigma_2^0 = F_\sigma$$

$$\Pi_2^0 = G_\delta$$

$$\Sigma_3^0 = G_{\delta\sigma}$$

$$\Pi_3^0 = F_{\sigma\delta}.$$

Note that, a set A is Borel if and only if there exists $\alpha < \omega_1$ such that A lies in Σ_α^0. Moreover, for each countable ordinal $\alpha < \omega_1$ another class in the hierarchy is defined

$$\Delta_\alpha^0 = \Sigma_\alpha^0 \cap \Pi_\alpha^0.$$

In particular

$$\Delta_1^0 = \text{all clopen sets.}$$

A subset $P \subseteq X$ of a Polish space is called *analytic* if it is the continuous image of a Borel set, i.e. there is $B \subseteq Y$ a Borel subset of a Polish space Y and $f : Y \to X$ continuous, with $f(B) = P$.

The complements of analytic sets are called *coanalytic*.

Therefore, we can continue our hierarchy by

$$\Sigma_1^1 = \text{all analytic sets}$$

$$\Pi_1^1 = \text{all coanalytic sets.}$$

and

$$\Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$$

Proposition 1.10. *The following facts hold*

1. *The analytic sets are closed under countable unions and intersections, and Borel images and preimages.*

2. *The coanalytic sets are closed under countable unions and intersections and Borel preimages.*

3. *The Borel sets are closed under complements, countable unions and intersections, as well as Borel preimages.*

Of great importance for us it is the fact that there are analytic (or coanalytic) non-Borel sets. We will see many examples of such sets later, but there are a couple of other ones occurring in analysis: the set of differentiable functions in $C([0,1])$ is Π^1_1 but not Borel (see [12]), and so is the set of functions in $C(\mathbb{T})$ with everywhere convergent Fourier series (see [1]).

We are in the position to see the main results in this topic.

Lemma 1.11. *Let (X,Θ) be Polish and $F \subseteq X$ closed. Let Θ_F be the topology generated by $\Theta \cup \{F\}$; i.e., the topology with basis $\Theta \cup \{U \cap F \ : \ U \in \Theta\}$. Then Θ_F is Polish, F is clopen in Θ_F, and $\mathcal{B}(\Theta_F) = \mathcal{B}(\Theta)$.*

Proof. Easy. □

Lemma 1.12. *Let (X,Θ) be Polish and let $(\Theta_n)_n$ be a sequence of Polish topologies on X with $\Theta \subseteq \Theta_n$, for each $n \in \omega$. Then the topology Θ_∞ generated by $\bigcup_n \Theta_n$ is Polish. Moreover, if $\Theta_n \in \mathcal{B}(\Theta)$, then $\mathcal{B}(\Theta_\infty) = \mathcal{B}(\Theta)$.*

Proof. Let $X_n = X$ for $n \in \omega$. Consider the map $\varphi : X \longrightarrow \prod_n X_n$ given by

$$\varphi(x) = (x, x, \ldots).$$

Note first that $\varphi(X)$ is closed in $\prod_n (X_n, \Theta_n)$. Indeed, if $(x_n) \notin \varphi(X)$, then for some $i < j$, $x_i \neq x_j$, so let U, V be disjoint open in Θ (thus also open in Θ_i, Θ_j resp.) such that $x_i \in U$, $x_j \in V$. Then

$$(x_n)_n \in X_0 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times \cdots \times X_{j-1} \times V \times X_{j+1} \times \cdots$$

where the right hand side is in the complement of $\varphi(X)$.

So $\varphi(X)$ is Polish. But φ is a homeomorphism of (X, Θ_∞) with $\varphi(X)$, so (X, Θ_∞) is Polish.

If $\Theta_n \in \mathcal{B}(\Theta)$ and $\{U_i^{(n)}\}_{i \in \omega}$ is a basis for Θ_n, then $\{U_i^{(n)}\}_{i,n \in \omega}$ is a subbasis for Θ_∞, so $\Theta_\infty \subseteq \mathcal{B}(\Theta)$ as well. □

Consider now the class $\mathcal{S}$ of subsets A of X for which there exists a Polish topology $\Theta_A \supseteq \Theta$ with $\mathcal{B}(\Theta_A) = \mathcal{B}(\Theta)$ and A clopen in Θ_A.

Let us show that $\Theta \subseteq \mathcal{S}$ and $\mathcal{S}$ is a σ-algebra. The first asserion follows by 1.11. Of course, $\mathcal{S}$ is closed under complements. Finally, let $\{A_n\} \subseteq \mathcal{S}$.

Let $\Theta_n = \Theta_{A_n}$ satisfy the above condition for A_n. Let Θ_∞ be as in 1.12. Then $A = \cup_n A_n$ is open in Θ_∞ and one more application of 1.11 completes the proof.

As consequence, we have the following fundamental fact about Borel sets in Polish spaces.

Theorem 1.13. *Let (X, Θ) be a Polish space and $A \subseteq X$ a Borel set. Then there is a Polish topology $\Theta_A \supseteq \Theta$ such that $\mathcal{B}(\Theta_A) = \mathcal{B}(\Theta)$ and A is clopen in Θ_A.*

Theorem 1.14. *(Lusin-Souslin)*
Let X be Polish and $A \subseteq X$ be Borel. There is a closed set $F \subseteq \omega^\omega$, and a continuous bijection $f : F \to A$. In particular, if $A \neq \emptyset$, there is also a continuous surjection $\widetilde{f} : \omega^\omega \to A$ extending f.

Proof. Enlarge the topology Θ of X to a Polish topology Θ_A in which A is clopen, thus Polish. By 1.3, there is a closed set $\mathcal{F} \subseteq \omega^\omega$ and a bijection $f : \mathcal{F} \longrightarrow A$ continuous for $\Theta_A|A$. Since $\Theta \subseteq \Theta_A$, $f : \mathcal{F} \longrightarrow A$ is continuous for Θ as well. □

The last Theorem precisely means that

$$\mathcal{B}(X) \subseteq \Sigma^1_1.$$

The next result tell us that implication is strict.

Theorem 1.15. *(Souslin)*
Let X be an uncountable Polish space. Then there exists always an analytic set of X which is not Borel.

Proof. Let Γ be a class of sets in arbitrary Polish spaces. By $\Gamma(X)$ we denote the subsets of X in Γ. If $\mathcal{U} \subseteq \omega^\omega \times X$, we call $\mathcal{U}$ *ω^ω-universal for* $\Gamma(X)$ if $\mathcal{U}$ is in $\Gamma(\omega^\omega \times X)$ and $\Gamma(X) = \{\mathcal{U}_y \ : \ y \in \omega^\omega\}$.

First notice that there exists a ω^ω-universal set for $\Sigma^0_1(\omega^\omega)$. Indeed, enumerate $\omega^{<\omega}$ in a sequence $(s_n)_n$ and put

$$(y, x) \in \mathcal{U} \iff x \in \bigcup\{\mathcal{N}_{s_i} \ : \ y(i) = 0\}$$

Since $\omega^\omega \times \omega^\omega$ is homeomorphic to ω^ω, it follows that there is an ω^ω-universal set for $\Sigma^0_1(\omega^\omega \times \omega^\omega)$, and by taking complements there is an ω^ω-universal set $\mathcal{F}$ for $\prod^0_1(\omega^\omega \times \omega^\omega)$. We now claim that $\mathcal{A} = \{(y, x) : \exists z, \ (y, x, z) \in \mathcal{F}\}$ is ω^ω-universal for $\Sigma^1_1(\omega^\omega)$. Since projection is continuous, $\mathcal{A}$ and all sections $\mathcal{A}_y$ are Σ^1_1. Conversely, if $A \subseteq \omega^\omega$ is Σ^1_1, there is closed $F \subseteq \omega^\omega$ and continuous surjection $f : F \to A$ (F could be empty). Let $\mathcal{G} = graph f^{-1}$ so that $\mathcal{G}$ is closed in $\omega^\omega \times \omega^\omega$ and

$$x \in A \iff \exists z \ : \ (x, z) \in \mathcal{G}.$$

Let $y \in \omega^\omega$ such that $\mathcal{G} = \mathcal{F}_y$. Then $A = \mathcal{A}_y$.

Now, $\mathcal{A}$ cannot be Borel. Indeed, in such case, $\mathcal{A}^c$ would be Borel too, so $A = \{x \ : \ (x, x) \notin \mathcal{A}\}$ would be Borel and thus analytic. So, for some y_0, $A = \mathcal{A}_{y_0}$, i.e.

$$(x, x) \notin \mathcal{A} \iff (y_0, x) \in \mathcal{A}.$$

Take $x = y_0$ to get a contradiction.

Since every uncountable Polish space X contains a homeomorphic copy of ω^ω (see Remark 1.8), it follows that there is an analytic set of X which is not Borel as well. □

The following result is of fundamental importance.

Theorem 1.16. *(The Lusin Separation Theorem)*
Let X be a Polish space and let $A, B \subseteq X$ be two disjoint analytic sets. Then there is a Borel set $C \subseteq X$ separating A from B, i.e., $A \subseteq C$ and $C \cap B = \emptyset$.

Proof. Assuming , without loss of generality, that A, B are nonempty, let $f : \omega^\omega \longrightarrow A$, $g : \omega^\omega \longrightarrow B$ be continuous surjections.

Put $A_s = f(\mathcal{N}_s), B_s = g(\mathcal{N}_s)$. Then

$$A_s = \bigcup_m A_{s^\frown m}, \quad B_s = \bigcup_n B_{s^\frown n}.$$

Suppose that $A_{s^\frown m}$, $B_{s^\frown m}$, for each $m, n \in \omega$ are Borel-separating, i.e. there exists $R_{m,n}$ Borel separating $A_{s^\frown m}$, $B_{s^\frown m}$, then $R = \cup_m \cap_n R_{m,n}$ separates A_s, B_s.

If A, B are not Borel-separating then, for what we have said, we can recursively define $x(n),\ y(n) \in \omega$ such that $A_{x|n},\ B_{y|n}$ are not Borel-separating for each $n \in \omega$. Then $f(x) \in A,\ g(y) \in B$, so $f(x) \neq g(y)$. Let $U,\ V$ be disjoint open sets with $f(x) \in U,\ g(y) \in V$. By the continuity of f and g, if n is large enough we have $f(\mathcal{N}_{x|n}) \subseteq U,\ g(\mathcal{N}_{y|n}) \subseteq V$, so U separates $A_{x|n}$ from $B_{y|n}$, a contradiction. □

Corollary 1.17. *Let X be a Polish space and $(A_n)_n$ be a pairwise disjoint sequence of analytic sets. Then there are pairwise disjoint Borel sets B_n, with $B_n \supseteq A_n$.*

As consequence, we have a celebrate theorem

Theorem 1.18. *(Souslin's Theorem)*
If X is a Polish space, then

$$\mathcal{B}(X) = \Delta^1_1(X)$$

Proof. Suppose that $A \in \Delta^1_1(X)$, then A and A^c are two disjoint analytic sets of X. It is enough to separate them by a Borel. □

It is now immediate the following

Proposition 1.19. *$f : X \longrightarrow Y$ has Borel graph if and only if f is Borel.*

Proof. For $V \subseteq Y$ open, we have

$$\begin{aligned} f(x) \in V &\Longleftrightarrow \exists y\ [f(x) = y \text{ and } y \in V] \\ &\Longleftrightarrow \forall y\ [f(x) = y \Longrightarrow y \in V] \end{aligned}$$

□

One of the many important consequences of the Suslin and Lusin Theorems is that the $1-1$ image of a Borel set by a Borel function is also Borel, i.e.

Theorem 1.20. *If B is a Borel subset of a Polish space X, $f : X \longrightarrow Y$ is a Borel map from X into a Polish space Y and f is $1-1$ on B, then $C = f(B)$ is also Borel.*

Proof. By 1.14, we can assume that $X = \omega^\omega$ and B closed.

Let $B_s = f(B \cap \mathcal{N}_s)$ for $s \in \omega^{<\omega}$. Since $f|_B$ is injective, we have

1. $B_\emptyset = B$;
2. $B_s = \bigcup_n B_{s^\frown n}$;
3. $B_{s^\frown i} \cap B_{s^\frown j} = \emptyset$, if $s \in \omega^\omega$ and $i \neq j$;
4. B_s is analytic.

By 1.17, we can find Borel sets B'_s such that

$$B'_\emptyset = Y \quad \text{and} \quad B_s \subseteq B'_s$$

We finally define by induction on $|s|$, Borel sets B^*_s such that

(a) $B^*_\emptyset = B'_\emptyset = Y$;

(b) $B^*_{s^\frown i} \cap B^*_{s^\frown j} = \emptyset$, if $s \in \omega^\omega$ and $i \neq j$;

(c) $B^*_{(n_0)} = B'_{(n_0)} \cap \overline{B_{(n_0)}}$;

(d) $B^*_{(n_0,\ldots,n_k)} = B'_{(n_0,\ldots,n_k)} \cap B^*_{(n_0,\ldots,n_{k-1})} \cap \overline{B_{(n_0,\ldots,n_k)}}$.

Then it is not hard to show by induction on k that

$$B_{(n_0,\ldots,n_k)} \subseteq B^*_{(n_0,\ldots,n_k)} \subseteq \overline{B_{(n_0,\ldots,n_k)}}.$$

We claim that

$$f(B) = \bigcap_k \bigcup_{s \in \omega^k} B^*_s$$

which shows that $f(B)$ is Borel.

If $x \in f(B)$, let $a \in B$ such that $f(a) = x$, so that $x \in \bigcap_k B_{a|k}$. Thus we have $x \in \bigcap_k B^*_{a|k}$.

Conversely, suppose $x \in \bigcap_k \bigcup_{s \in \omega^k} B^*_s$, then there is a unique $a \in \omega^\omega$ such that $x \in \bigcap_k B^*_{a|k}$. Then also $x \in \bigcap_k \overline{B_{a|k}}$; so, in particular, $B_{a|k} \neq \emptyset$ for all k and thus $B \cap \mathcal{N}_{a|k} \neq \emptyset$ for all k. Which means that $a \in B$, since B is closed. So $f(a) \in \bigcap_k B_{a|k}$.

We claim that $f(a) = x$. Otherwise, since f is continuous, there is a neighborhood $\mathcal{N}_{a|k_0}$ of a with $f(\mathcal{N}_{a|k_0}) \subseteq U$, where U is an open such that $x \notin \overline{U}$. Then

$$x \notin \overline{f(\mathcal{N}_{a|k_0})} \supseteq \overline{B_{a|k_0}},$$

a contradiction. □

The following theorem will be very useful in the future

Theorem 1.21. *(Representation Theorem for Analytic Set) Let X be a Polish space and let $P \subseteq X$ be an analytic set. Then there is a closed set $F \subseteq X \times \omega^\omega$ such that*

$$x \in P \iff \exists \epsilon \in \omega^\omega \ : \ (x, \epsilon) \in F;$$

i.e. every Σ^1_1-set is the projection of a closed set in $X \times \omega^\omega$ (the converse is true by definition).

Also, there is a G_δ-set $G \subseteq X \times 2^\omega$ such that

$$x \in P \iff \exists \epsilon \in 2^\omega \ : \ (x, \epsilon) \in G;$$

(One cannot replace here G_δ by closed, since 2^ω is compact and projection of compact set are compact).

Proof. By standard facts,

every non-empty analytic set is the continuous image of ω^ω

So, let $f : \omega^\omega \longrightarrow X$ be continuous such that $f(\omega^\omega) = P$ (provided $P \neq \emptyset$, since otherwise the result is trivial). Then let us define $F \subseteq X \times \omega^\omega$ by

$$(x, \epsilon) \in F \Longleftrightarrow f(\epsilon) = x.$$

For the second statement notice that ω^ω can be identify with a G_δ subset of 2^ω as follows. Let

$$\langle \cdot, \cdot \rangle : \omega \times \omega \longrightarrow \omega$$

be a $1-1$ correspondence and assign to each $\epsilon \in \omega^\omega$ the element $\epsilon^* \in 2^\omega$ given by

$$\epsilon^*(\langle n, m \rangle) = 0 \quad \text{if and only if} \quad \epsilon(n) = m.$$

The map $\epsilon \longmapsto \epsilon^*$ is a homomorphism between ω^ω and the following G_δ subset of 2^ω

$$\{\delta \in 2^\omega \ : \ \text{for all } n \text{ there is a unique } m \text{ with } \delta(\langle n, m \rangle) = 0\}$$

□

For the future, we will need the following tools

Definition 1.22. Let X be a Polish space.

1. A subset P of X is called Π^1_1-*hard* if for any Polish space Y and any $Q \in \Pi^1_1(Y)$ there exists a Borel function $f : Y \longrightarrow X$ such that $Q = f^{-1}(P)$.
2. A subset P of X is called Π^1_1-*complete* if P is Π^1_1-hard and $P \in \Pi^1_1(X)$.

Remark 1.23. Note that **any Π^1_1-hard subset of a Polish space, is not analytic, thus not Borel**.

Indeed, if there is a Π^1_1-hard subset of a Polish space which is also analytic, then every coanalityc set of any Polish space should be Borel preimage of an analytic set. By Proposition 1.10, any coanalytic set should be also analytic, and then (by Souslin's Theorem 1.18) Borel. But the last assertion contradict Theorem 1.15.

A *measurable space* is a pair (X, S) where X is a set and S is a σ-algebra on X.

A measurable space (X, S) is said to be a *standard Borel space* if there exists a Polish topology τ on X such that the Borel σ-algebra of (X, τ) coincide with the σ-algebra S.

A basic example of standard Borel space is the *Effros-Borel structure.* Precisely, for every Polish space X by $F(X)$ we denote the set of all closed subsets of X. We endow $F(X)$ with the σ-algebra S generated by the family

$$\{\ \{F \in F(X)\ :\ F \cap U \neq \emptyset\}\ :\ U \text{ is an open subset of } X\}$$

The measurable space $(F(X), S)$ is called the Effros-Borel space of $F(X)$.

Before to enunciate the next theorem, let us recall that, if X is a topological space, on $K(X)$ (the space of all compact subsetes of X) we equip the *Vietoris topology*, i.e., the one generated by the sets of the form

$$\begin{gathered}\{K \in K(X)\ :\ K \subseteq U\}\\ \{K \in K(X)\ :\ K \cap U \neq \emptyset\}\end{gathered}$$

Notice that, if (X, d) is a metric space the *Hausdorff metric* on $K(X)$, defined by

$$d_H(K, L) = \begin{cases} 0, & \text{if } K = L = \emptyset; \\ 1, & \text{if exactly on of } K, L \text{ is } \emptyset; \\ \max\{\max_{x \in K} d(x, L)\ ,\ \max_{x \in L} d(x, K)\} & \text{if } K, L \neq \emptyset. \end{cases}$$

Then the Hausdorff metric is compatible with the Vietoris topology. Moreover, if D is a countable dense of X, then $K_f(D) = \{K \subseteq D\ :\ K \text{ is finite}\}$ is also a countable dense of $K(X)$. We have showed the following

Proposition 1.24. *If X is a Polish space, so is $K(X)$.*

Theorem 1.25. *If X is a Polish, then the Effros-Borel space of $F(X)$ is standard.*

Proof. Let $\overline{X}$ be a compactification of X. Then the map

$$F \in F(X) \longmapsto \overline{F} \in K(\overline{X}),$$

where $\overline{F}$ denotes the closure of F in $\overline{X}$ is injective, since $F = \overline{F} \cap X$. We claim that $G = \{\overline{F}\ :\ F \in F(X)\}$ is G_δ in $K(\overline{X})$.

Indeed, for $K \in K(\overline{X})$, $K \in G \Leftrightarrow K \cap X$ is dense in K.

If $X = \bigcap_n U_n$, where U_n is open in $\overline{X}$, and letting $\{V_m\}_m$ a basis for $\overline{X}$, we have by the Baire category Theorem

$$K \in G \iff \forall n\ (K \cap U_n \text{ is dense in } K)$$

$$\iff \forall n\, \forall m\, (K \cap U_m \neq \emptyset \Rightarrow K \cap (V_m \cap U_n) \neq \emptyset).$$

Thus G is Polish.

Transfer back to $F(X)$ its topology via the bijection $F \longmapsto \overline{F}$ to get a Polish topology $\mathcal{T}$ on $F(X)$. We have to verify that the Borel space of this topology is the Effros-Borel space.

It is easy to verify that the sets $\{K \in K(\overline{X}) \ : \ K \cap U \neq \emptyset\}$, for U open in $\overline{X}$, generate $\mathcal{B}(K(\overline{X}))$ respect to $\mathcal{T}$. But

$$\{F \in F(X) \ : \ \overline{F} \cap U \neq \emptyset\} = \{F \in F(X) \ : \ F \cap (U \cap X) \neq \emptyset\}$$

so these are exactly the generators of the Effros-Borel structure. □

Let us also recall the following useful result (see also [18, Theorem 5.2.1]).

Theorem 1.26. *(Kuratowski and Ryll-Nardzewski Selection Theorem for $F(X)$)*
Let X and Y be Polish spaces and $F : Y \longrightarrow F(X)$ a Borel map such that $F(y) \neq \emptyset$ for every $y \in Y$. Then there exists a sequence $f_n : Y \longrightarrow X$ of Borel selectors of F (i.e. $f_n(y) \in F(y)$ for every $n \in \omega$ and $y \in Y$) such that the sequence $(f_n(y))_n$ is dense in $F(y)$ for all $y \in Y$.

Corollary 1.27. *Let X be a Polish space. There is a sequence of Borel functions $d_n : F(X) \longrightarrow X$, such that for every $F \in F(X)$, $\{d_n(F)\}_n$ is dense in F.*

Proof. Since X is Polish, by Theorem 1.3 there exists a continuous (and open) surjection $f : \omega^\omega \longrightarrow X$. Let $\{U_s\}_{s \in \omega^{<\omega}}$ the open balls that played in the construction of f.

For a given nonempty $F \in F(X)$, let

$$T_F = \{s \in \omega^{<\omega} \ : \ F \cap U_s \neq \emptyset\}$$

and note that T_F is a nonempty tree with the property that every $s \in T_F$ has a proper extension $t \supsetneq s$, $t \in T_F$ (a tree with such property is said *pruned*).

In particular $[T_F] \neq \emptyset$. We define the *leftmost branch* of T_F denoted by a_{T_F} by

$$a_{T_F}(n) = \text{ the least element } m \text{ of } \omega \text{ such that } [(T_F)_{(a_T|n)^\frown m}] \neq \emptyset$$

Let $d(F) = f(a_F)$ so that $d(F) \in F$. Define also $d(\emptyset) = x_0$, some fixed element in X.

Now the function $g : F(X) \setminus \{\emptyset\} \longrightarrow \omega^\omega$ given by $g(F) = a_F$ is Borel. Indeed, given a basic open set $\mathcal{N}_s$, $s \in \omega^n$, we have

$$g(F) \in \mathcal{N}_s \iff F \cap U_s \neq \emptyset \text{ and } \forall t \in \omega^n \ (t <_{lex} s \Rightarrow F \cap U_t = \emptyset),$$

where $<_{lex}$ is the lexicographical ordering on ω^n. So d is Borel too.

Fix now a basis $\{V_n\}_n$ of nonempty open set in X. By the above argument, we can find, for each n, a Borel function $d'_n : F(X) \longrightarrow X$ such that $d'_n(F) \in F \cap V_n$ if $F \cap V_n \neq \emptyset$. Finally, let

$$d_n(F) = \begin{cases} d'_n(F), & \text{if } F \cap V_n \neq \emptyset; \\ d(F), & \text{if } F \cap V_n = \emptyset. \end{cases}$$

□

Sembra che tutte le grandi cose abbiano bisogno, per iscriversi con eterne esigenze nel cuore dell'umanità, di passare prima sulla terra sotto la forma di maschere mostruose e terrificanti ...

F. Nietzsche
Al di là del bene e del male

Chapter 2

Few Facts about Trees

Let Λ be a set. With $\Lambda^{<\omega}$ we denote the set of all finite sequence of elements of Λ, and with Λ^ω the set of all infinite sequence of elements of Λ. Of course, if Λ^n is the set of all sequence of Λ with length n, then

$$\Lambda^{<\omega} = \bigcup_n \Lambda^n$$

If $s,\ t \in \Lambda^{<\omega}$ we denote by

$$s \subseteq t$$

the extension order of finite sequence, if $|s| \leq |t|$ (length of s less than length of t), and for each $i \leq |s|$, $s(i) = t(i)$. In this case we say also that s is an initial segment of t and write

$$s = t|m \quad \text{with } m = |s|.$$

Note that $\emptyset \subseteq s$, for all s. Similarly if $\epsilon \in \Lambda^\omega$ is an infinite sequence we write

$$s \subseteq \epsilon \quad \text{or} \quad s = \epsilon|m$$

where $m = |s|$, if $s = (\epsilon(0), \ldots, \epsilon(m-1))$.

Let T be a subset of $\Lambda^{<\omega}$. We say that T is a *tree* if

$$t \in T \text{ and } s \subseteq t \quad \text{implies } s \in T$$

For a tree T on Λ, a *branch* through T is an $\varepsilon \in \Lambda^\omega$ such that for all $n \in \omega$,

$$\varepsilon|n = (\varepsilon(0), ..., \varepsilon(n-1)) \in T.$$

We denote by

$$[\theta] = \{\varepsilon \in \omega^\omega \ : \ \varepsilon \text{ is a branch through } \theta\}$$

The *body* of θ.

We call θ *well founded* if $[\theta] = \emptyset$, i.e. θ has no branches. Otherwise, we will call θ *ill founded*.

A first example of tree is ω; it is a tree on ω. In this case $[\omega] = \{\omega\}$.

Another simply example of tree is $\omega^{<\omega}$. This last tree, the tree of all finite sequence of ω will be denote by T, and the set of all trees on ω, i.e. the set of subtrees of T, is denoted by $\mathcal{T}$. A tree $\theta \in \mathcal{T}$ can be identified with its characteristic function which is a member of the Polish space $\{0,1\}^{\omega^{<\omega}} = 2^{\omega^{<\omega}}$ homeomorphic to 2^ω (i.e. the Cantor set). The set of all trees is then easily a closed subset of $2^{\omega^{<\omega}}$. The following is a classical fact.

Theorem 2.1. *Let $\mathcal{WF}$ be the set of well founded trees on $\mathcal{T}$. Then $\mathcal{WF}$ is a Π^1_1-complete set (in $2^{\omega^{<\omega}}$).*

Proof. Let X be a fixed Polish space and $P \in \Pi^1_1(X)$. By 1.21, let $F \subseteq X \times \omega^\omega$ be a closed such that

$$x \notin P \Longleftrightarrow \exists \epsilon \in \omega^\omega \ \ (x, \epsilon) \in F$$

Assign now to each $x \in X$ the tree

$$T(x) = \{s \in \omega^{<\omega} \ : \ \exists V \text{ open in } X \text{ with } x \in V,\ diam V \leq 2^{-|s|} \text{ and } (V \times \mathcal{N}_s) \cap F \neq \emptyset\}$$

Then $\{x \in X \ \ s \in T(x)\}$ is open in X, for all $s \in \omega^{<\omega}$. Therefore the function

$$x \stackrel{f}{\longmapsto} T(x)$$

is a Borel from X into $2^{\omega^{<\omega}}$.

We claim now that

$$(*) \qquad x \notin P \Longleftrightarrow T(x) \notin \mathcal{WF},$$

thus $P = f^{-1}(\mathcal{WF})$, so $\mathcal{WF}$ is Π^1_1-complete.

Indeed, if $x \notin P$, let $\epsilon \in \omega^\omega$ such that $(x, \epsilon) \in F$. Then ϵ is a branch through $T(x)$. Conversely, if ϵ is a branch through $T(x)$ let, for each $n \in \omega$, $\epsilon_n \in \omega^\omega$ and $x_n \in X$ such that

$$\epsilon_n | n = \epsilon | n, \quad (x_n, \epsilon_n) \in F \qquad \text{and} \quad d(x_n, x) \leq 2^{-n}.$$

Then $x_n \to x$, $\epsilon_n \to \epsilon$; so $(x, \epsilon) \in F$ and by construction $x \notin P$.

To see that $\mathcal{WF}$ is Π^1_1 notice that

$$T \in \mathcal{WF} \Longleftrightarrow \forall \sigma \in \omega^\omega \ \exists k \in \omega \text{ with } \sigma | k \notin T.$$

□

A direct consequence of Remark 1.23 is the following

Corollary 2.2. *$\mathcal{WF}$ is coanalytic but not Borel in the Polish space $\mathcal{T}$ equipped with the topology inherited from $2^{\omega^{<\omega}}$.*

Let us also recall the *hight* of a tree.

Let Λ be an infinite set and let $\kappa = |\Lambda|$. For every well founded tree T on Λ we define

$$T' = \{s \in T \ : \ \exists t \in T \text{ with } s \subsetneqq t\} \in \mathcal{WF}(\Lambda).$$

By transfinite recursion, we define the iterated derivatives T^ξ, $\xi < \kappa^+$ by

$$T^0 = T, \quad T^{\xi+1} = (T^\xi)' \quad \text{and} \quad T^\lambda = \bigcap_{\xi<\lambda} T^\xi \quad \text{if } \lambda \text{ is limit}$$

Notice that if $T^\xi \neq \emptyset$, then $T^{\xi+1} \subsetneqq T^\xi$. It follows that the transfinite sequence $(T^\xi)_{\xi<\kappa^+}$ is eventually empty (see 4.1 below). The hight $ht(T)$ of T is defined to be the least ordinal ξ such that $T^\xi = \emptyset$. If T is ill founded, then by convention we set $ht(T) = \kappa^+$. In particular, if Λ is countable, then $ht(T) < \omega_1$ for every $T \in \mathcal{WF}(\Lambda)$ while $ht(T) = \omega_1$ for every T ill founded on Λ.

Let S and T be tree on Λ_1 and Λ_2 respectively. A map $\phi : S \longrightarrow T$ is called *monotone* if for every $s_1,\ s_2 \in S$ with $s_1 \subsetneqq s_2$ we have $\phi(s_1) \subsetneqq \phi(s_2)$. The following fact is quite useful.

Proposition 2.3. *Let S and T be trees on ω. Then $ht(S) \leq ht(T)$ if and only if there exists a monotone map $\phi : S \longrightarrow T$.*

Proof. Suppose a such monotone map exists. If T is ill founded, then obviously we have $ht(S) \leq ht(T)$. So, assume that T is well founded. We see that for every countable ordinal ξ and every $s \in S^\xi$ we have $\phi(s) \in T^\xi$. That would implies $ht(S) \leq ht(T)$.

Conversely, assume that $ht(S) \leq ht(T)$. If T is ill founded, then choose $\sigma \in [T]$. For every $s \in S$ we let $\phi(s) = \sigma|_{|s|}$. It is clear that ϕ is monotone.

Suppose now that T is well founded, we construct ϕ as follows:

we set $\phi(\emptyset) = \emptyset$. Let $k \in \omega$ and assume we have already defined $\phi(s)$ for all $s \in S$ with $|s| \leq k$ so that

$$(*) \qquad \forall \xi < \omega_1, \quad s \in S^\xi \Rightarrow \phi(s) \in T^\xi$$

Let $w \in S$ with $|w| = k+1$. Then there exists $s \in S$ with $|s| = k$, $n \in \omega$ such that $w = s \frown (n)$. Let $t = \phi(s)$. By construction, we see that there exists $m \in \omega$ such that, setting $\phi(w) = t \frown m$, property $(*)$ is satisfied for w and $\phi(w)$. It is easily seen that ϕ is monotone. □

Siamo un segno che non ha significato, siamo senza dolore, e abbiamo quasi perso il linguaggio in terra straniera.

F. Hölderlin
Liriche

Chapter 3

Rank Theory

Definable ranks are fundamental tolls in Descriptive Set Theory. We will treat some property of ranked classes valid for Π^1_1. Before to define the main tool of this section, we need to recall the following

Definition 3.1. Let X be a set and $\prec$ a relation on X (i.e. $\prec \subseteq X \times X$). We say that $\prec$ is <u>*well founded*</u> if every nonempty subset $Y \subseteq X$ has a $\prec$-minimal element.

This is equivalent to assert that there is no infinite decreasing chain.

By recursion, we can define the <u>*rank function*</u>

$$\rho_\prec : X \longrightarrow ORD \qquad \text{(where } ORD \text{ is the class of ordinals)}$$

given by

$$\rho_\prec(x) = \sup\{\rho_\prec(y) + 1 \;\; y \prec x\}$$

In particular $\rho_\prec(x) = 0$ if x is minimal. Note also that $\rho_\prec$ maps X onto some ordinal α, which is $< card(X)^+$. Let us define the rank of $\prec$ by

$$\rho(\prec) = \sup\{\rho_\prec(x) + 1 \; : \; x \in X\}$$

Theorem 3.2. *Let X be a Polish space and $\prec$ an analytic well founded relation on X. Then $\rho(\prec)$ is countable.*

Proof. We can clearly assume that $X = \omega^\omega$. Let us define the tree $T_\prec$ on ω^ω associate with $\prec$

$$(x_0, \ldots, x_{n-1}) \in T_\prec \Longleftrightarrow x_{n-1} \prec \ldots \prec x_1 \prec x_0$$

Of course, $(x) \in T_\prec$ for all $x \in X$.

Since $\prec$ is well founded, it is easy to see that the associate tree $T_\prec$ is well founded as well, and $\rho(\prec) = ht(T_\prec)$. So it is enough to show that $ht(T_\prec) < \omega_1$. This will be done by proving that there exist a countable set W, a well founded relation $\prec_*$ on W and an order preserving map from $(T_\prec \setminus \{\emptyset\}, \supsetneqq)$ into $(W, \prec_*)$.

Let S be a pruned tree on $\omega \times \omega \times \omega$ such that

$$x \prec y \iff \exists z \text{ with } (x, y, z) \in [S],$$

see [7, Proposition I.2.4].

Let W be defined as the set of all sequences of the form

$$w = ((s_0, t_0, u_0), \ldots, (s_{n-1}, t_{n-1}, u_{n-1}))$$

where $(s_i, t_i, u_i) \in S$ and $s_i = t_{i+1}$ for all $i < n - 1$.

Let us define $\prec_*$ on W by

$$w \prec_* w' \iff \begin{array}{l} |w| < |w'| \\ \text{and} \\ \forall i < |w| \; (s'_i, t'_i, u'_i) \supsetneqq (s_i, t_i, u_i). \end{array}$$

We claim that the relation $\prec_*$ is well founded.

Otherwise, let $w_n = \big((s_0^n, t_0^n, u_0^n), \ldots, (s_{k_n-1}^n, t_{k_n-1}^n, u_{k_n-1}^n)\big)$ be such that $w_{n+1} \prec_* w_n$. Then $k_n \uparrow \infty$. Letting $l_n = ht(s_i^n)$ $(= ht(t_i^n) = ht(u_i^n)$, for $i < k_n)$, also $l_n \uparrow \infty$, and there are $x_0, x_1, \ldots$ in ω^ω and $z_0, z_1, \ldots$ in ω^ω such that for all n,

$$t_0^n \subseteq x_0, \; s_0^n = t_1^n \subseteq x_1, \; s_1^n = t_2^n \subseteq x_2, \; \ldots \quad \text{and} \quad u_0^n \subseteq z_0, \; u_1^n \subseteq z_1, \; \ldots$$

Thus $(x_1, x_0, z_0) \in [S]$, $(x_2, x_1, z_1) \in [S]$, $\ldots$; i.e. that is $x_1 \prec x_0$, $x_2 \prec x_1$, $\ldots$, which is a contradiction.

Now, notice that if $x \prec y$, the section tree

$$S(x, y) = \{s \in \omega^{<\omega} \; : \; (x|_{|s|}, y|_{|s|}, s) \in S\}$$

is not well founded, so let $h_{x,y} \in [S(x, y)]$.

Finally, let us define

$$f : T_\prec \setminus \{\emptyset\} \longrightarrow W$$

given by

$$f((x)) = \emptyset$$

and for $n \geq 2$,

$$f(x_0,\ldots,x_{n-1}) = \big((x_1|n, x_0|n, h_{x_1,x_0}|n),\ldots,(x_{n-1}|n, x_{n-2}|n, h_{x_{n-1},x_{n-2}}|n)\big)$$

Then $f(x_0,\ldots,x_{n-1},x_n) \prec_* f(x_0,\ldots,x_{n-1})$ for any $n \geq 1$, so our proof is complete. □

Definition 3.3. Let X be a Polish space and B be a Π^1_1 subset of X. A map $\phi : B \longrightarrow ORD$ is said to be a Π^1_1-*rank on* B if there are relations $\leq_\Sigma$, $\leq_\Pi \subseteq X \times X$ in Σ^1_1 and Π^1_1 respectively, such that for every $y \in B$ we have

$$\begin{aligned}\phi(x) \leq \phi(y) &\Longleftrightarrow x \leq_\Sigma y\\ &\Longleftrightarrow x \leq_\Pi y\end{aligned}$$

Here we have some basic property of Π^1_1-rank

Theorem 3.4. *Let X be a Polish space, B a Π^1_1 subset of X and $\phi : B \longrightarrow ORD$ a Π^1_1-rank on B. Let denote by $\alpha = \phi(B)$. Then the following hold*

(0) $\alpha \leq \omega_1$;

(i) For every countable ordinal ξ the set $B_\xi = \{x \in B \ : \ \phi(x) \leq \xi\}$ is Borel;

(ii) B is Borel if and only if $\sup\{\phi(x) \ \ x \in B\} < \omega_1$;

(iii) For every analytic subset A of B, we have $\sup\{\phi(x) \ \ x \in A\} < \omega_1$.

Proof. (i) Follows from Theorem 1.18 (i.e. $\Delta^1_1(X) = \mathcal{B}(X)$).

(0) For $x \in B$, let us define the relation

$$y \prec z \Longleftrightarrow \begin{array}{l}\phi(y) \leq \phi(x)\\ \text{and}\\ \phi(z) \leq \phi(x)\\ \text{and}\\ \phi(y) < \phi(z).\end{array}$$

Then the relation $\prec$ is Borel and well founded, so $\phi(x) = \rho(\prec) < \omega_1$ by Theorem 3.2. So $\alpha \leq \omega_1$.

(ii) If B is Borel, then the relation

$$y \prec' z \Longleftrightarrow \ \ y,\ z \in B \ \text{ and } \ \phi(y) < \phi(z)$$

is Borel and $\rho(\prec') = \sup\{\phi(x) \ x \in B\} < \omega_1$. The converse is clear.

(iii) Suppose that $\sup\{\phi(x) \ x \in A\} = \omega_1$, then by (ii) B cannot be Borel. We show that there is $x_0 \in B$ such that $\phi(x) \leq \phi(x_0)$ for all $x \in A$, and by (0) we get a contradiction.

Indeed, if not

$$\begin{aligned} x \in B \iff & \ \exists y \in A \ \text{ and } \ \phi(x) \leq \phi(y) \\ \iff & \ \exists y \in A \ \text{ and } \ x \leq_\Sigma y. \end{aligned}$$

Therefore B should be also analytic. By Theorem 1.18 we get B Borel (OOPS!!!). □

The next Proposition shows that Π^1_1-rank can be defined only using two analytic relations

Proposition 3.5. *Let X be a Polish space, B a Π^1_1 subset of X and $\phi : B \longrightarrow \omega_1$.*

Then ϕ is a Π^1_1-rank on B if and only if there are relations $\leq'_\Sigma$, $<'_\Sigma \subseteq X \times X$ both in Σ^1_1 such that for every $x \in B$ we have

$$\phi(x) \leq \phi(y) \iff \ x \leq'_\Sigma y$$

and

$$\phi(x) < \phi(y) \iff \ x <'_\Sigma y$$

Proof. It is enough to note that, if $\leq_\Sigma$, $\leq_\Pi$ are the analytic and coanalytic relation in the definition of Π^1_1-rank, we have just to choose $\leq'_\Sigma = \leq_\Sigma$ and

$$x <'_\Sigma \iff \ x \leq_\Sigma \ \text{ and not } \ y \leq_\Pi x$$

□

Let us see a first important example of Π^1_1-rank.

Theorem 3.6. *The map $ht : \mathcal{WF} \longrightarrow \omega_1$ defined by*

$$T \longmapsto ht(T)$$

is a Π^1_1-rank on $\mathcal{WF}$.

Proof. We have already seen in Theorem 2.1 that $\mathcal{WF}$ is Π^1_1.

We define

$$S \leq_\Sigma T \iff T \notin \mathcal{WF} \text{ or}$$

$$S, T \in \mathcal{WF} \text{ and } ht(S) \leq ht(T)$$

and

$$S <_\Sigma T \iff T \notin \mathcal{WF} \text{ or } S, T \in \mathcal{WF} \text{ and } ht(S) < ht(T)$$

By Proposition 2.3

$$S \leq_\Sigma T \iff \exists f : S \to T \text{ monotone}$$

and so the relation $\leq_\Sigma$ is Σ^1_1.

For every $T \in \mathcal{T}$ and every $\lambda \in \omega$, we set

$$T_\lambda = \{t \in T \ : \ (\lambda) \frown t \in T\}$$

Observe that if $T \in \mathcal{WF}$ then $ht(T) = \sup\{ht(T_\lambda) \ : \ \lambda \in \omega\}$, while if T is ill founded, then there exists $\lambda \in \omega$ such that T_λ is also ill founded. Invoking again Proposition 2.3, we have

$$S <_\Sigma T \iff \exists \lambda \in \omega \text{ and } \exists f : S \to T_\lambda \text{ monotone.}$$

Then $<_\Sigma$ is also Σ^1_1. □

Definition 3.7. Let X and Y be Polish spaces, $A \subseteq X$ and $B \subseteq Y$. We say that A is *(resp.Borel)-reducible* to B if there exists a continuous (resp. Borel) map $f : X \longrightarrow Y$ such that $f^{-1}(B) = A$.

The following it is easy to proof

Lemma 3.8. *Let X and Y be Polish spaces, $A \subseteq X$ and $B \subseteq Y$. Assume that A is Borel reducible to B via the Borel map $f : X \longrightarrow Y$. Assume, moreover, that B is Π^1_1 and $\phi : B \longrightarrow \omega_1$ is a Π^1_1-rank on B. Then A is Π^1_1 and the map $\psi : A \longrightarrow \omega_1$ defined by $\psi(x) = \phi(f(x))$ is a Π^1_1-rank on A.*

Theorem 3.9. *Let Λ be a countable set and A be a subset of Λ^ω. Then A is Σ^1_1 if and only if there exists a tree T on $\Lambda \times \omega$ such that*

$$A = p[T] = \{\sigma \in \Lambda^\omega \ : \ \exists \tau \in \omega^\omega \text{ with } (\sigma, \tau) \in [T]\}.$$

Proof. See [7]. □

The following is fundamental in Rank Theory.

Theorem 3.10. *Let X be a Polish space and $B \subseteq X$ be a Π^1_1 set. Then there exists a Π^1_1-rank on B.*

Proof. By the previous lemma, it is enough to find Borel reduction of B to $\mathcal{WF}$. Of course, we can suppose that $X = \omega^\omega$. By Theorem 3.9, there exists a tree T on $\omega \times \omega$ such that $B^c = p[T]$. For every $\sigma \in \omega^\omega$ we let

$$T(\sigma) = \{t \in \omega^{<\omega} \ : \ (\sigma|_{|t|}, t) \in T\} \in \mathcal{T}.$$

It is easy to see that the map $f : \omega^\omega \longrightarrow \mathcal{T}$ defined by $f(\sigma) = T(\sigma)$ is continuous. To finish, observe that

$$\begin{aligned} \sigma \notin B &\iff \exists \tau \in \omega^\omega \text{ with } (\sigma, \tau) \in [T] \\ &\iff T(\sigma) \text{ is ill founded} \end{aligned}$$

and so $f^{-1}(\mathcal{WF}) = B$.

□

Immagine può essere anzitutto la veduta di un determinato ente, in quanto manifesto nella sua semplice-presenza. Questo ente offre una veduta *anblick*. In senso derivato, “immagine” può inoltre significare sia la veduta che ricalca una semplice presenza (copia) o, meglio, riproduce un ente che non è piú presente, sia la veduta che prospetta un ente ancora da produrre.

M. Heidegger
Kant e il problema della metafisica

Chapter 4

Derivatives

Let us start with the following general observation

Theorem 4.1. *Let X be a second countable topological space and $(F_\alpha)_{\alpha<\rho}$ a strictly increasing or decreasing transfinite sequence of closed (or open) set; i.e. $\alpha < \beta \Longrightarrow F_\alpha \supsetneqq F_\beta$.*

Then ρ is a countable ordinal.

Proof. Let us suppose that $(F_\alpha)_{\alpha<\rho}$ is a strictly decreasing transfinite sequence of closed set. Let $\{U_n\}_n$ be an open basis for X.

To each closed set $F \subseteq X$ let us associate the set of numbers

$$N(F) = \{n \in \omega \ : \ U_n \cap F \neq \emptyset\}$$

It is enough to note that the map

$$F \longmapsto N(F)$$

is injective and increasing (i.e., $F \subseteq G \Longrightarrow N(F) \subseteq N(G)$). Therefore, a strictly decreasing transfinite sequence of closed $(F_\alpha)_{\alpha<\rho}$, it will produce a strictly decreasing transfinite sequence of subsets of ω. Thus, ρ has to be countable. □

A typical example is the Cantor-Bendixson derivative on the set of all compact subset of X, denoted by $K(X)$. Precisely, for $K \in K(X)$ Cantor-Bendixson derivative of K is

$$K' = \{x \in K \ : \ x \text{ is a limit point of } K\}$$

More in general,

Definition 4.2. Let X be a Polish space. A map $D : K(X) \longrightarrow K(X)$ is said to be a *derivative* on $K(X)$ if

(i) $D(K) \subseteq K$, for all $K \in K(X)$;

(ii) $D(K_1) \subseteq D(K_2)$, whenever $K_1,\ K_2 \in K(X)$ with $K_1 \subseteq K_2$.

Given a derivative D and $K \in K(X)$ by transitive recursion we can define the iterated derivatives on K by

$$D^0(K) = K,\ D^{\xi+1}(K) = D(D^{\xi}(K)) \quad \text{and} \quad D^{\lambda}(K) = \bigcap_{\xi<\lambda} D^{\xi}(K) \text{ if } \lambda \text{ is a limit ordinal}$$

Clearly, $(D^{\xi}(K))_{\xi<\omega_1}$ is a transfinite decreasing sequence of compact subsets of X, and so, it is eventually constant. The *D-rank* of K, denoted by $|K|_D$ is defined to be the least ordinal ξ such that $D^{\xi}(K) = D^{\xi+1}(K)$. For simplicity of notation, in the sequel we will use $D^{|K|_D}(K) = D^{\infty}(K)$.

Given X and Y two Polish space, a map $\mathbb{D} : Y \times K(X) \longrightarrow K(X)$ is said to be a *parameterized derivate* if for any $y \in Y$ the map

$$\mathbb{D}_y : K(X) \longrightarrow K(X)$$

defined by

$$\mathbb{D}_y(K) = \mathbb{D}(y, K)$$

is a derivative on $K(X)$.

The following result, due to Kechris and Woodin (see [8]), it is really useful.

Theorem 4.3. *Let X and Y be Polish spaces and $\mathbb{D} : Y \times K(X) \longrightarrow K(X)$ be a parameterized derivative. Assume that $\mathbb{D}$ is Borel. Then the set*

$$\Omega_{\mathbb{D}} = \{(y, K) \in Y \times K(X)\ :\ \mathbb{D}_y^{\infty}(K) = \emptyset\}$$

is a Π_1^1 set and the map

$$(y,\ K) \longmapsto |K|_{\mathbb{D}_y}$$

is a Π_1^1-rank on $\Omega_{\mathbb{D}}$.

For the future, it will be useful the following variation

Theorem 4.4. *Let X be a Polish space and $D_n : K(X) \longrightarrow K(X)$ be a sequence of Borel derivatives on $K(X)$. Then the set*

$$\Omega = \{K \in K(X)\ :\ D_n^{\infty}(K) = \emptyset,\ \forall n \in \omega\}$$

is Π_1^1 set and the map

$$K \longmapsto \sup\{|K|_{D_n}\ :\ n \in \omega\}$$

is a Π_1^1-rank on Ω.

Proof. Let $n \in \omega$ arbitrary. Let us apply Theorem 4.3 for $Y = \{n\}$ and $\mathbb{D} = D_n$, we get that the set $\Omega_{D_n} = \{K \in K(X) \ : \ D_n^\infty(K) = \emptyset\}$ is Π^1_1.

Since $\Omega = \bigcap_n \Omega_{D_n}$, we get that Ω is Π^1_1 too.

For sake of notation, we set $\phi(K) = \sup\{|K|_{D_n} \ : \ n \in \omega\}$ for every $K \in K(X)$.

Let $Y = \omega$ with the discrete topology and consider the map

$$\mathbb{D} : Y \times K(X) \longrightarrow K(X)$$

defined by

$$\mathbb{D}(y, K) = D_n(K)$$

Then $\mathbb{D}$ is a parameterized Borel derivative. By Theorem 4.3, the map

$$(n, K) \longmapsto |K|_{\mathbb{D}_n} = |K|_{D_n}$$

is a Π^1_1-rank on $\Omega_{\mathbb{D}}$. Let $\leq_\Sigma$ and $\leq_\Pi$ be the associate relations. For every $K \in K(X)$ we have

$$\begin{aligned}(H \in \Omega) \text{ and } \phi(H) \leq \phi(K) &\Longleftrightarrow \forall n \in \omega \ \exists m \in \omega \text{ with } (n, H) \leq_\Sigma (m, K)\\ &\Longleftrightarrow \forall n \in \omega \ \exists m \in \omega \text{ with } (n, H) \leq_\Pi (m, K).\end{aligned}$$

Hence ϕ is Π^1_1-rank on Ω and the proof is completed. □

We close this section by mentioning the following result concerning sets in product spaces with compact sections. Although it is not related to the notion of a Π^1_1-rank, it is very useful tool for checking that various derivatives are Borel.

Theorem 4.5. *Let X and Y be Polish spaces and $A \subseteq Y \times X$ be such that for every $y \in Y$ the section $A_y = \{x \in X \ : \ (y, x) \in A\}$ of A at y is compact. Consider the map $\Phi_A : Y \longrightarrow K(X)$ defined by*

$$\Phi_A(y) = A_y.$$

Then the set A is Borel if and only if Φ_A is a Borel map.

To show that Theorem, we will need the following

Lemma 4.6. *(Kunugui, Novikov)*
Let X and Y be Polish spaces and $A \subseteq Y \times X$ a Borel be such that every section A_y is open. Then if $\{U_n\}_n$ is any open basis for X,

$$A = \bigcup_n (B_n \times U_n),$$

with B_n Borel in Y.

Proof. If $(y, x) \in A$, then for some $n \in \omega$, $x \in U_n \subseteq A_y$. So $A = \bigcup_n (Y_n \times U_n)$, where

$$Y_n = \{y \in Y \ : \ U_n \subseteq A_y\}$$

Clearly Y_n is a Π^1_1 set. If $Z_n = Y_n \times U_n$, then Z_n is Π^1_1, and $A = \bigcup_n Z_n$. By Lusin separation theorem (see Corollary 1.17), there is a sequence $(A_n)_n$ of Borel sets with $A = \bigcup_n A_n$ and $A_n \subseteq Z_n$.

Let $S_n = proj_Y(A_n) \subseteq Y_n$. Then S_n is Σ^1_1, so by Lusin separation theorem (Theorem 1.16, applied to S_n and Y_n^c), there is a Borel set B_n with

$$S_n \subseteq B_n \subseteq Y_n.$$

Then $A_n \subseteq B_n \times U_n \subseteq Y_n \times U_n = Z_n$, and so

$$A = \bigcup_n (B_n \times U_n).$$

□

Proof. (*of Theorem 4.5*)
We can first assume that X is compact, by replacing it by a compactification if necessary. By Kunugui-Novikov's Lemma,

$$A^c = \bigcup_n (B_n \times U_n),$$

where $\{U_n\}_n$ is a open basis of X and $B_n \subseteq Y$ is Borel. Thus

$$y \in A_y \Longleftrightarrow \forall n \ (y \in B_n \Rightarrow x \notin U_n).$$

Put $X \setminus U_n = K_n$, $b(y) = \{n \in \omega \ : \ y \in B_n\}$.

Then $b : Y \longrightarrow 2^\omega$ is Borel and $A_y = \bigcap_{n \in b(y)} K_n$.

The proof that $y \longmapsto A_y$ is Borel will be finished one we prove the following

Claim The map $S \longmapsto \bigcap_{n \in S} K_n$, from 2^ω into $K(X)$ is Borel.

It is enough to show that if $F \subseteq X$ is closed, then

$$P = \{S \in 2^\omega \ : \ \bigcap_{n \in S} K_n \cap F \neq \emptyset\}$$

is Borel. Let us define

$$(S, x) \in R \Longleftrightarrow \ \forall n \in \omega \ (n \in S \Rightarrow x \in K_n) \text{ and } x \in F$$

Then $R \subseteq 2^\omega \times X$ is closed, so compact. Moreover, $P = proj_{2^\omega}(R)$ is compact too. To finish the Claim, it is enough to apply Kuratowski-Ryll-Nardzewski's Theorem (see Theorem 1.27). □

Il piacere di vedere ci sprofonda nell'emozione della presenza che ci allontana dal movimento del significato ... lo sguardo si lacera fra presenza e sapere, ma da questa lacerazione nasce il pensiero.

B. Noël
Journal du regard

Chapter 5

Banach Space Theory: a glimpse

Recall that, a *linear space* X is a *normed space* if to each $x \in X$ corresponds a real number $\|x\|$, called the *norm* of x, which satisfies the conditions:

(i) $||x|| \geq 0$ for each $x \in X$ and $\|x\| = 0$ iff $x = 0$;

(ii) $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$, for every α scalar and $x \in X$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.

A *complete normed space* X is called a *Banach space*.

For two Banach spaces X, Y, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y (i.e., all maps which are linear and continuous respect the norm topologies of X and Y). It is easy to see that this last space become a Banach space if endowed with the following norm

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X \leq 1} \|T(x)\|_Y.$$

In particular, if $Y = \mathbb{K}$, the scalar field, we denote by $\mathcal{L}(X, \mathbb{K}) = X^*$, usually called the *dual space* of X.

For each $x^* \in X^*$ let $D_{x^*} = \mathbb{K}$, and let $\mathcal{D} = \prod_{x^* \in X^*} D_{x^*}$. Let $T : X \longrightarrow \mathcal{D}$ the map defined by

$$T(x) = (x^*(x))_{x^* \in X^*}.$$

Then T is one-to-one embedding of X into $\mathcal{D}$. The *weak topology* on X is defined as the topology induced by $\mathcal{D}$ via the map T. Similarly, we can define on X^* a weaker topology, called the *weak* topology*, which is induced by $\widetilde{\mathcal{D}} = \prod_{x \in X} D_x$, where $D_x = \mathbb{K}$, for each $x \in X$. It is classical, and easy to prove, that the closed unit ball B_{X^*} of X^* is weak* compact (in the literature such a result is called the *Banach-Alaoglu-Boubaki*).

Definition 5.1. Let X be a Banach space. Then X is called:

reflexive if the natural embedding $\imath : X \hookrightarrow X^{**}$ given by

$$\imath(x)(x^*) = x^*(x)$$

is a isometric isomorphism.

with the *Shur property* if any weakly convergent sequence in X is norm convergent (i.e., weak topology and norm topology coincide sequentially).

uniformly convex if for each $0 < \varepsilon < 2$ there exists a $\delta(\varepsilon) > 0$ such that whenever $x, y \in X$ with $\|x\|,\ \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, then

$$\|\frac{x+y}{2}\| \leq 1 - \delta(\varepsilon).$$

has the *Krein Milman property* if for each closed bounded convex subset C of the unit ball B_X of X has an extreme point (remember that an element $x \in C$ is said to be an *extreme point* of C if whenever there are $x_1,\ x;2 \in C$ such that $x = \lambda x_1 + (1 - \lambda)x_2$ for some $0 \leq \lambda \leq 1$, then $x = x_1 = x_2$).

has the *Radon Nikodym property* if whenever given a probability space (Ω, Σ, μ) and a countable additive, μ-continuous measure $F : \Sigma \longrightarrow X$ of bounded variation, there is a Bochner integrable $f : \Omega \longrightarrow X$ such that for each $E \in \Sigma$, we have

$$F(E) = \int_E f(w) d\mu(w)$$

About the Radon Nikodym property, it is classic the following

Theorem 5.2. *Let X be a Banach spaces. Then TFAE*

(a) X has the Radon Nikodym property;

(b) Given a probability space (Ω, Σ, μ) and a vector measure $G : \Sigma \longrightarrow X$ such that $\|G(E)\| \leq \mu(E)$, for each $E \in \Sigma$, there is a (necessarily essentially bounded) Bochner integrable $g : \Omega \longrightarrow X$ such that for any $E \in \Sigma$,

$$G(E) = \int_E g(\omega) d\mu(\omega);$$

(c) *Given a bounded linear operator $T : L_1[0,1] \longrightarrow X$, there is a (necessarily essentially bounded) Bochner integrable $h : [0,1] \longrightarrow X$ such that for any $f \in L_1[0,1]$,*

$$Tf = \int_{[0,1]} f(t)\ h(t)\ dt;$$

(d) *Any uniformly bounded martingale sequence (X_n, Σ_n) having values in X is almost surely convergent;*

(e) *every non void closed bounded convex subset C of X is dentable, i.e. given such a C and any $\epsilon > 0$, there is an $x_\epsilon \in C$ such that*

$$x_\epsilon \notin \overline{co}(C \setminus \{y \in C : \|y - x\| < \epsilon\});$$

(f) *Every non void closed bounded convex subset C of X has a denting point, i.e. given such a C, there is a point $x \in C$ (called a denting point) such that regardless $\epsilon > 0$*

$$x \notin \overline{co}(C \setminus \{y \in C : \|y - x\| < \epsilon\});$$

(g) *Every non void closed bounded convex subset C of X is the closed convex hull of its denting points.*

Recall that a sequence $(x_n)_n$ in a Banach space X is said to be a *Schauder basis* of X if for every $x \in X$ there exists a unique sequence $(\alpha_n)_n$ of scalars such that $x = \sum_{n\in\omega} \alpha_n x_n$, where we are considering the convergence of the series in the norm topology of X. A sequence $(x_n)_n$ is called a *basic sequence* if it is a Schauder basis for $\overline{span}\{x_n \ : \ n \in \omega\}$ (the closed linear span of $(x_n)_n$, i.e. the smallest closed linear subspace containing the sequence).

For a Schauder basis $(x_n)_n$ we denote by $(x_n^*)_n$ the sequence of *bi-orthogonal functionals* associate to $(x_n)_n$. That means

$$x_m^*(x_n) = \delta(n,m) = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{otheowise.} \end{cases}$$

For a subset $F \subseteq \omega$, let us denote by $P_F : X \longrightarrow \overline{span}\{x_n \ : \ n \in F\}$ the projection defined by: if $x = \sum_{n\in\omega} \alpha_n x_n$ then $P_F(x) = \sum_{n\in F} \alpha_n x_n$. The *basis constant* of $(x_n)_n$ is defined as the number $\sup\{\|P_{\{0,\ldots,n\}}\| \ : \ n \in \omega\}$. If $x = \sum_{n\in\omega} \alpha_n x_n$, the *support* of x is defined to be the set $\{n \in \omega \ : \ \alpha_n \neq 0\}$.

Definition 5.3. Let $(x_n)_n$ be a Schauder basis of a Banach space X and $C \geq 1$.

(i) The basis $(x_n)_n$ is said to be *monotone* if its basis constant is 1. It is said to be *bi-monotone* if $\|P_I\| = 1$ for every interval I of ω;

(ii) The basis $(x_n)_n$ is said to be C*-unconditional* if $\|P_F\| \leq C$ for every subset F of ω. The basis $(x_n)_n$ is said to be *unconditional* if it is C-unconditional for some $C \geq 1$;

(iii) The basis $(x_n)_n$ is said to be *shrinking* if the sequence $(x_n^*)_n$ of bi-orthogonal functionals associate to $(x_n)_n$ is a Schauder basis of X^*;

(iv) The basis $(x_n)_n$ is said to be *boundedly complete* if for each sequence $(\alpha_n)_n$ of scalars such that $\sup_{k\in\omega} \| \sum_{n=0}^{k} \alpha_n x_n\| < \infty$ we have that the series $\sum_{n\in\omega} \alpha_n x_n$ converges;

(v) A sequence $(v_k)_k$ in X is said to be *block* respect to the basis, if $\max\{n \in \omega \ : \ n \in supp(v_k)\} < \max\{n \in \omega \ : \ n \in supp(v_{k+1})\}$.

Two sequence $(x_n)_n$ and $(y_n)_n$, in two Banach space X and Y, respectively, are said to be C*-equivalent* (in the sequel we shall use the notation $(x_n)_n \sim_C (y_n)_n$), where $C \geq 1$, if for every $k \in \omega$ and every $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ we have

$$\frac{1}{C} \ \|\sum_{n=0}^{k} \alpha_n y_n\|_Y \leq \|\sum_{n=0}^{k} \alpha_n x_n\|_X \leq C \ \|\sum_{n=0}^{k} \alpha_n y_n\|_Y$$

.

The following result asserts that basic sequence are invariant under small perturbations.

Proposition 5.4. *Let X be a Banach space and $(x_n)_n$ be a semi-normalized basic sequence in X with basis constant $K \geq 1$. If If $(y_n)_{n=0}^{l}$ is a finite sequence in X such that*

$$\|x_n - y_n\| \leq \frac{1}{2K} \cdot \frac{1}{2^{n+2}}$$

for every $n \in \{0, \ldots, l\}$, then $(y_n)_{n=0}^{l}$ is 2-equivalent to $(x_n)_{n=0}^{l}$

Let us recall that two Banach space X and Y are said to be *isometric* if there exists a bounded linear operator $T \in \mathcal{L}(X, Y)$ such that $\|T(x)\| = \|x\|$ for each $x \ inX$.

Now we are ready to enunciate two fundamental universality results in Banach Space Theory.

Theorem 5.5. *Let X be a separable Banach space. Then there exists a closed subspace Y of $C(2^\omega)$ which is isometric to X.*

Proof. Let $K = (B_{X^*}, \text{weak}^*)$ be the unit ball of the dual space X^* endowed with the weak* topology. Since X is separable, then K is a compact metrizable space, where the metric that induce the weak* topology on B_{X^*} is the following

$$d(x^*, y^*) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \, |x^*(x_n) - y^*(x_n)|$$

where $(x_n)_n$ is a dense sequence in the unite ball of X.

By Theorem 1.4 there exists a continuous surjection $f : 2^\omega \longrightarrow K$. Let us define

$$T : X \longrightarrow C(2^\omega)$$

by

$$T(x)(\sigma) = f(\sigma)(x), \quad \text{for every } \sigma \in 2^\omega,\ x \in X.$$

It is easy to check that T is a linear isometric embedding. □

Before to go to the other universal space Theorem, we recall some notation:

for $1 \leq p \leq \infty$ and a sequence of Banach spaces $(X_n)_n$, we denote by $\left(\bigoplus_{n\in\omega} X_n\right)_{\ell_p}$ (or simply by $\left(\bigoplus_{n\in\omega} X_n\right)_p$), the Banach space of all sequence $(x_n)_n$ such that

$x_n \in X_n$ for all $n \in \omega$;

$\left[\sum_n \|x_n\|_{X_n}^p\right]^{\frac{1}{p}} < \infty$.

The second result is due to A. Pelczynski (see [14] or [16])

Theorem 5.6. *There exists a space $\mathcal{U}$ with a normalized bi-monotone Schauder basis $(u_n)_n$ such that for every seminormalized basic sequence $(x_n)_n$ in a Banach space X there exists $L = \{l_0 < l_1 < \cdots\} \in [\omega]$ such that $(x_n)_n$ is equivalent to $(u_{l_n})_n$ and the natural projection P_L onto $\overline{span}\{u_n : n \in L\}$ has norm one. Moreover, if U' is another space with the above properties, then U' is isomorphic to $\mathcal{U}$.*

Proof. Let $(d_n)_n$ be a countable dense subset of the sphere of $C(2^\omega)$, and let $(x_n)_n$ be a seminormalized basic sequence in a Banach space X. By Theorem 5.5 and Proposition 5.4 there exists $L = \{l_0 < l_1 < \cdots\} \in [\omega]$ such that $(x_n)_n$ is equivalent to $(d_{l_n})_n$. Now, let us construct the space $\mathcal{U}$.

Let Σ denotes the tree on ω consisting of all nonempty strictly increasing finite sequence. For $t = (n_0 < \ldots < n_k) \in \Sigma$, we let $n_t = n_k$. Fix a bijection $\varphi : \Sigma \longrightarrow \omega$ such that

$$\varphi(t) < \varphi(s) \Leftrightarrow t \subsetneqq s.$$

For every $t \in \Sigma$ we define $f_t \in C(2^\omega)$ by

$$f_t = d_{n_t}.$$

Let us define $\mathcal{U}$ the completion of $c_{00}(\Sigma)$, space of all sequences (indexed on Σ) with finite support, under the norm

$$\|x\|_{\mathcal{U}} = \sup\left\{ \left\| \sum_{t\in\mathfrak{s}} x(t) f_t \right\|_{C(2^\omega)} \ : \ \mathfrak{s} \text{ is a segment of } \Sigma \right\}.$$

According to φ, let $(u_n)_n$ be the enumeration of the standard basis $(e_t)_{t\in\Sigma}$ of $c_{00}(\Sigma)$, i.e.

$$e_t(s) = \begin{cases} 1, & \text{if } s = t \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $(u_n)_n$ defines a normalized bi-monotone Schauder basis of $\mathcal{U}$.

For every $\sigma \in [\Sigma]$ we set $L_\sigma = \{\varphi(\sigma|k) \ : \ k \geq 1\} \in [\omega]$. If $\{l_0 < l_1 < \cdots\}$ is the increasing enumeration of L_σ, then we set $X_\sigma = \overline{span}\{u_{l_n} \ : \ n \in \omega\}$.

Let $P_\sigma : \mathcal{U} \longrightarrow X_\sigma$ be the natural projection. Notice that $\|P_\sigma\| = 1$. By the assertion at the beginning of the proof, we see that for every seminormalized basic sequence $(x_n)_n$ in a Banach space X, there exists $\sigma \in [\Sigma]$ such that if $L_\sigma = \{l_0 < l_1 < \cdots\}$, then $(x_n)_n$ is equivalent to $(u_{l_n})_n$. Hence the space $\mathcal{U}$ has the desired properties.

Suppose that U' is another space with the properties described in the statement of the theorem. Therefore, there are Banach space X and Y such that

$$\mathcal{U} = U' \oplus X \quad \text{and} \quad U' = \mathcal{U} \oplus Y.$$

Moreover there exists another space Z such that

$$\mathcal{U} = (\mathcal{U} \oplus \mathcal{U} \oplus \cdots)_{\ell_2} \oplus Z$$

Thus

$$\mathcal{U} \oplus \mathcal{U} \cong \mathcal{U} \oplus (\mathcal{U} \oplus \mathcal{U} \oplus \cdots)_{\ell_2} \oplus Z \cong (\mathcal{U} \oplus \mathcal{U} \oplus \cdots)_{\ell_2} \oplus Z \cong \mathcal{U}$$

Similarly we have that $U' = U' \oplus U'$. To finish, it is enough to note that

$$\mathcal{U} \cong U' \oplus X \cong U' \oplus U' \oplus X \cong U' \oplus \mathcal{U} \cong \mathcal{U} \oplus \mathcal{U} \oplus Y \cong \mathcal{U} \oplus Y \cong U'.$$

□

L'*ira* é designata diversamente in latino, in greco e ovunque, in ragione della diversitá delle lingue. Ma l'espressione del viso di un uomo in collera non é nè latina nè greca. Se qualcuno dice *iratus sum*, nessun popolo, tranne il latino, lo capisce. Ma se la collera della sua anima esacerbata gli sale al viso e ne modifica l'espressione, tutti i presenti dicono "Ecco un uomo in collera".

Agostino
De doctrina Christiana

Chapter 6

Families of Banach Spaces

In the following chapter, we are going to describe some useful technique introduced by B. Bossard in [2].

Let X be a separable Banach space. We endow the set $F(X)$ of all closed subsets of X with the Effros-Borel structure. By Corollary 1.27 there exists a sequence $d_n : F(X) \longrightarrow X$ of Borel maps such that

(i) $d_n(F) \in F$ for every $F \in F(X)$ and $n \in \omega$;

(ii) $(d_n(F))_n$ is dense in F, for every $F \in F(X)$.

Notice that, a closed subspace F is a linear subspace of X if and only if

$$(0 \in F) \text{ and } (\forall n, m \in \omega,\ \forall p, q \in \mathbb{Q} \text{ we have} \tag{6.1}$$

$$p\ d_n(F) + q\ d_m(F) \in F).$$

It is easy to see that (6.1) defines a Borel subset of $F(X)$. By Theorem 5.5 the space $C(2^\omega)$ is isometrically universal for all separable Banach spaces.

For a separable Banach space X we denote by $\mathcal{SE}(X)$ the subset of $F(X)$ consisting of the closed subspace of X. We abbreviate $\mathcal{SE}(C(2^\omega))$ by $\mathcal{SE}$. Using this notation, we have already proved that

Proposition 6.1. *Let X be a separable Banach space. Then $\mathcal{SE}(X)$ is a Borel subset of $F(X)$ equipped with the Effros-Borel structure.*

Let us start to the construction of two auxiliaries Banach spaces following, in the spirit, to the James tree space (see [6] or [11]).

We denote by $c_{00}(T)$ the space of finitely supported function from $T = \omega^{<\omega}$ to $\mathbb{R}$ and by $\chi_s : T \longrightarrow \{0, 1\}$ the characteristic function of $\{s\}$ for every $s \in T$. Thus $c_{00}(T) = span\{\chi_s\ :\ s \in T\}$.

An *admissible choice of intervals* is a finite set $\{I_j \ : \ 0 \leq j \leq k\}$ of interval of T such that every branch of T meets at most one of these intervals.

We define the following norms on $c_{00}(T)$:

$$\|y\|_1 = \sup \left(\sum_{j=0}^{k} \left\| \sum_{s \in I_j} y(s) \ u_{|s|} \right\|_{\mathcal{U}} \right)$$

$$\|y\|_2 = \sup \left[\sum_{j=0}^{k} \left\| \sum_{s \in I_j} y(s) \ u_{|s|} \right\|_{\mathcal{U}}^2 \right]^{\frac{1}{2}}$$

where the supremum is taken over $k \in \omega$ and over all admissible choice of intervals $\{I_j \ : \ 0 \leq j \leq k\}$.

Then we let $U_1(T)$ to be the completion of $c_{00}(T)$ under the norm $\|\cdot\|_1$ and $U_2(T)$ to be the completion of $c_{00}(T)$ under the norm $\|\cdot\|_2$. In the sequel, for $A \subseteq \omega^{<\omega}$, we denote by $U_1(A)$ (resp. $U_2(A)$) the closed subspace of $U_1(T)$ (resp. $U_2(T)$) generated by $\{\chi_s \ : \ s \in A\}$.

We are going to give several lemmas useful to understand the structure of the spaces introduced above.

Lemma 6.2. *The sequence $\{\chi_{s_i} \ : \ i \in \omega\}$ determines a basis for $U_1(T)$ and $U_2(T)$. For any $A \subseteq T$, $\{\chi_{s_i} \ : \ s_i \in A\}$ determines a basis for $U_1(A)$ and $U_2(A)$.*

Proof. We give the proof only for $U_2(T)$. For $U_1(T)$ follows similarly.

Let $(\lambda_i)_{i\in\omega}$ be a sequence in $\mathbb{R}$, I an interval of T and $n, p \in \omega$. Let us denote by $c_{\underline{u}}$ the basis constant for the universal basis $\underline{u} = (u_n)_n$ of $\mathcal{U}$.

Let $\mathcal{K} : \omega \longrightarrow \omega^{<\omega}$ be an enumeration of $\omega^{<\omega}$ such that if $s \subsetneqq t$ then $\overline{s} < \overline{t}$, where $\overline{s} = \mathcal{K}^{-1}(s)$.

For $s \in T$, $(\sum_{i=0}^{n} \lambda_i \chi_{s_i})(s)$ is equal to $\lambda_{\overline{s}}$ if $\overline{s} \leq n$, and 0 if not. Therefore

$$\begin{aligned}\|\sum_{s\in I}(\sum_{i=0}^{n} \lambda_i \chi_{s_i})(s) \ u_{|s|}\|_{\mathcal{U}} &= \|\sum_{\substack{s\in I \\ \overline{s}\leq n}} \lambda_{\overline{s}} u_{|s|}\|_{\mathcal{U}} \leq c_{\underline{u}} \ \|\sum_{\substack{s\in I \\ \overline{s}\leq n+p}} \lambda_{\overline{s}} u_{|s|}\|_{\mathcal{U}} \\ &= c_{\underline{u}} \ \|\sum_{s\in I}(\sum_{i=0}^{n+p} \lambda_i \chi_{s_i})(s) \ u_{|s|}\|_{\mathcal{U}}\end{aligned}$$

since for $s, t \in I$, then $t \supseteqq s$ if and only if $\overline{t} \geq \overline{s}$.

Let $\{I_j \ : \ 0 \leq j \leq k\}$ be an admissible choice of intervals. We have

$$\sum_{j=0}^{k} \| \sum_{s\in I_j} (\sum_{i=0}^{n} \lambda_i \chi_{s_i})(s)\ u_{|s|}\|_{\mathcal{U}}^2 \leq c_{\underline{u}}^2 \ \sum_{j=0}^{k} \| \sum_{s\in I} (\sum_{i=0}^{n+p} \lambda_i \chi_{s_i})(s)\ u_{|s|}\|_{\mathcal{U}}^2.$$

Thus $\| \sum_{i=0}^{n} \lambda_i \chi_{s_i} \|_2 \leq c_{\underline{u}} \ \| \sum_{i=0}^{n+p} \lambda_i \chi_{s_i} \|_2$ and $\{\chi_{s_i} : i \in \omega\}$ is a basic sequence. □

Lemma 6.3. *Let b be a branch of T. Then*

(i) The space $U_1(\{s : s \subsetneq b\} = U_1(b)$ and $U_2(b)$ are both isomorphic to $\mathcal{U}$.

(ii) If $\theta \in \mathcal{T}$ and if b is a branch of θ, then for $r \in \{1,2\}$ $U_r(b)$ is complemented in $U_r(\theta)$.

Proof. Let $r \in \{1,2\}$.

(i) Since $\{\chi_{b|j} : j \in \omega\}$ is a basis of $U_r(b)$, it is enough to prove that $\underline{u}$ (the universal basis) is equivalent to $\{\chi_{b|j} : j \in \omega\}$.

Let $(\lambda_j)_{j=0}^n \in \mathbb{R}^{<\omega}$. We have

$$\begin{aligned}\| \sum_{j=0}^{n} \lambda_j \chi_{b|j} \|_r &= \sup \left\{ \| \sum_{s\in I} (\sum_{j=0}^{n} \lambda_j \chi_{b|j})(s)\ u_{|s|} \| \ : \ I \text{ interval},\ I \subseteq \{s : s \subsetneq b\} \right\} \\ &= \sup\{ \| \sum_{j=l}^{m} \lambda_j u_j \| \ : \ 0 \leq l \leq m \leq n\}.\end{aligned}$$

Thus

$$\| \sum_{j=0}^{n} \lambda_j u_j \|_{\mathcal{U}} \leq \| \sum_{j=0}^{n} \lambda_j \chi_{b|j} \|_r \leq 2c_{\underline{u}} \ \| \sum_{j=0}^{n} \lambda_j u_j \|_{\mathcal{U}}$$

(ii) Let $y = \sum_{i\in\omega} y(s_i)\chi_{s_i}$ be an element of $U_r(\theta)$. We have

$$\begin{aligned}\| \sum_{\substack{i\in\omega \\ s_i\in b}} y(s_i)\chi_{s_i} \|_r &= \sup \left\{ \| \sum_{s\in I} y(s)\ u_{|s|} \| \ : \ I \text{ interval},\ I \subseteq \{s : s \subsetneq b\} \right\} \\ &\leq \|y\|_r\end{aligned}$$

□

Lemma 6.4. *Let $(A_i)_{i\in\omega}$ be a sequence of subsets of T such that every branch meets at most one of these subsets. Then for $r \in \{1,2\}$ the spaces*

$$U_r(\bigcup_{i\in\omega} A_i) \ \textit{and} \ (\bigoplus_{i\in\omega} U_r(A_i))_{\ell_r} \ \textit{are isometric}$$

Proof. We give the proof for $r = 2$. The other case follows similarly.

Pick $y \in span\left\{\chi_s \ : \ s \in \bigcup_{i\in\omega} A_i\right\}$. We let $y_i = \sum_{s\in A_i} y(s)\chi_s$. Since the set $\{y_i : i \in \omega \text{ and } y_i \neq 0\}$ is finite, there is $m \in \omega$ such that $y = \sum_{i=0}^m y_i$. To finish the proof, it is enough to show the following

Claim $\|y\|_2^2 = \sum_{i=0}^m \|y_i\|_2^2$.

Indeed, let $\{I_j \ : \ 0 \leq j \leq k\}$ be an admissible choice of intervals. We set, for $0 \leq j \leq k$ and $0 \leq i \leq m$, $I_j(y) = \sum_{s\in I_j} y(s)u_{|S|}$ and $M_i = \{j \in \omega : 0 \leq j \leq k, \ I_j \cap A_i \neq \emptyset\}$. The largest interval with ends in $I_j \cap A_i$ is denoted by J_j^i. For any $i \in \omega$, $\{J_j^i : j \in M_i\}$ is an admissible choice of intervals, thus

$$\sum_{j=0}^{k} \|I_j(y)\|^2 = \sum_{i=0}^{m} \sum_{j\in M_i} \|J_j^i(y_i)\|^2 \leq \sum_{i=0}^{m} \|y_i\|_2^2.$$

It follows by taking the supremum over admissible choices of intervals that

$$\|y\|_2^2 \leq \sum_{i=0}^{m} \|y_i\|_2^2.$$

Now, for any $0 \leq i \leq m$, let $\{I_j^i \ : \ 0 \leq j \leq_i\}$ be an admissible choice of intervals. We denote by $\widetilde{I}_j^i$ the largest interval with ends in $I_j^i \cap A_i$. Then $\{\widetilde{I}_j^i \ : \ 0 \leq i \leq m, \ 0 \leq j \leq k_i\}$ is an admissible choice of intervals, because every branch of T meets at most one of the A_i's. For any i

$$\sum_{j=0}^{k_i} \|I_j^i(y_i)\|^2 = \sum_{j=0}^{k_i} \|\widetilde{I}_j^i(y_i)\|^2 = \sum_{j=0}^{k_i} \|I_j^i(y)\|^2,$$

$$\sum_{i=0}^{m} \sum_{j=0}^{k_i} \|I_j^i(y_i)\|^2 = \sum_{i=0}^{m} \sum_{j=0}^{k_i} \|\widetilde{I}_j^i(y)\|^2 \leq \|y\|_2^2$$

thus

$$\sum_{i=0}^{m} \|y_i\|_2^2 \leq \|y\|_2^2.$$

□

The next lemma is classical for functional analysts, and it is left as simple exercise.

Lemma 6.5. *Let $(X_j)_{j\in\omega}$ be a sequence of Banach spaces.*

(i) if all X_j's is reflexive, then $(\bigoplus_{j\in\omega} X_j)_{\ell_2}$ is reflexive;

(ii) if all X_j's has the Schur property , then $(\bigoplus_{j\in\omega} X_j)_{\ell_1}$ has the Schur property.

The following result is really constructive.

Theorem 6.6. *Let $\theta \in \mathcal{T}$.*

(i) If θ is ill founded, then $U_1(\theta)$ and $U_2(\theta)$ are isomorphic to $\mathcal{U}$, thus universal;

(ii) If θ is well founded, then $U_2(\theta)$ is reflexive, and $U_1(\theta)$ has the Schur property.

Proof. (i) If θ is ill founded, we pick b a branch of θ. By Lemma 6.2 and Lemma 6.3, $U_1(\theta)$ and $U_2(\theta)$ are Banach space with a basis, which contain a complemented copy of $U(b) = \mathcal{U}$. Therefore, they contains a complemented copy of every Banach space with a basis. By Theorem 5.6 both spaces are isomorphic to $\mathcal{U}$.

(ii) For $\theta \in \mathcal{T}$, $s \in T$ and $i \in \omega$, we define

$$s \frown \theta = \{s \frown t : t \in \theta\}, \qquad \theta_i = \{t \in T : (i) \frown t \in \theta\}.$$

Since $U_r(\theta) = U_r(\emptyset \frown \theta)$, to prove the theorem, it is enough to show the following

Claim If θ is well founded, then for any $s \in T$, $U_1(s \frown \theta)$ has the Schur property, and $U_2(s \frown \theta)$ is reflexive.

We will show the Claim using transfinite induction on $ht(\theta)$.

We assume that for every tree $\tau \in \mathcal{T}$ such that $ht(\tau) < \alpha < \omega_1$, $U_1(s \frown \tau)$ has the Schur property and $U_2(s \frown \tau)$ is reflexive for any $s \in T$.

Let θ such that $ht(\theta) = \alpha$, and for $s \in T$ let $N_s = \{i \in \omega : s \frown (i) \in \theta\}$. We let $A_i = s \frown (i) \frown \theta_i$ for $i \in N_s$, so that $\cup_{i\in N_s} A_i = s \frown (\theta \setminus \{s\})$ and every branch of T meets at most one of the A_i's. If $i \in N_s$ then $ht(\theta_i) < \alpha$, thus $U_1(A_i)$ has the Schur property, and $U_2(A_i)$ is reflexive. By Lemma 6.4, we have

$$U_r(s \frown (\theta \setminus \{s\})) = U_r(\bigcup_{i\in N_s} A_i) = (\bigoplus_{i\in N_s} U_r(A_i))_{\ell_r},$$

thus by Lemma 6.5, $U_1(s \frown (\theta \setminus \{s\}))$ has the Schur property and $U_2(s \frown (\theta \setminus \{s\}))$ is reflexive.

Since $\{\chi_{s_j} : \ j \in \omega, \ s_j \in s \frown \theta\}$ is a basis of $U_r(s \frown \theta)$ with the first element χ_s and the other element generate $U_r(s \frown (\theta \setminus \{s\}))$. Then, we have that $U_r(s \frown \theta) \cong \mathbb{R} \times U_r(s \frown (\theta \setminus \{s\}))$. Thus $U_1(s \frown \theta)$ has the Schur property and $U_2(s \frown \theta)$ is reflexive. $\square$

Lemma 6.7. *The map* $\varphi : \mathcal{T} \longrightarrow \mathcal{SE}$ *defined by*

$$\varphi(\theta) = U_2(\theta)$$

is Borel.

Proof. Let O be an open subset of $C(2^\omega)$. It is enough to show that $\Omega = \{\theta \in \mathcal{T} : U_2(\theta) \cap O \neq \emptyset\}$ is Borel.

Since $\{\chi_{s_i} : \ i \in \omega, s_i \in \theta\}$ defines a basis of $U_2(\theta)$, we have

$$U_2(\theta) \cap O \neq \emptyset \Leftrightarrow \exists \lambda \in \mathbb{Q}^{<\omega} \text{ such that } \sum_{i=0}^{n} \lambda_i \chi_{s_i} \in O \text{ and if } \lambda_i \neq 0 \text{ then } s_i \in \theta.$$

Let $\Lambda = \{\lambda \in \mathbb{Q}^{<\omega} : \ \sum_{i=0}^{n} \lambda_i \chi_{s_i} \in O\}$. Then

$$\Omega = \bigcup_{\lambda \in \Lambda} \bigcap_{i \in supp(\lambda)} \{\theta \in \mathcal{T} : \ s_i \in \theta\}$$

thus Ω is Borel since $\{\theta \in \mathcal{T} : \ s_i \in \theta\}$ is an open and closed subset. □

The following are useful.

If X is a separable Banach space, we will denote by $\langle X \rangle$ the equivalent class $\{Y \in \mathcal{SE} \ : \ Y \cong X\}$ of the isomorphism relation $\cong$.

Proposition 6.8. *The class* $\langle \mathcal{U} \rangle$ *is not Borel and the relation* $\cong$ *is not Borel.*

Proof. Since $\varphi^{-1}(\langle \mathcal{U} \rangle) = \mathcal{IF}$ and $\mathcal{IF}$ is not Borel (see 2.2), it follows that $\langle \mathcal{U} \rangle$ is not a Borel class and consequently $\cong$ is not a Borel relation. □

The proof of the next lemma is left to the reader.

Lemma 6.9. *Let* X *be a separable Banach space.*

(i) $\{(F, y) \ : \ y \in F\}$ *is Borel in* $F(X) \times X$*;*

(ii) $\{(Y, \ (y_n)_n) \ : \ \overline{span}(y_n)_n = Y\}$ *is Borel in* $\mathcal{SE}(X) \times X^\omega$*;*

(iii) $\{(\ (x_n)_n, \ (y_n)_n \ ) \ : \ (x_n)_n \sim (y_n)_n\}$ *is Borel in* $X^\omega \times X^\omega$*;*

(iv) $\{(Y, Z) \ : \ Y \subseteq Z\}$ *is Borel in* $\mathcal{SE}(X) \times \mathcal{SE}(X)$*.*

Proposition 6.10. *The isomorphism relation* $\cong$ *is analytic in* $\mathcal{SE} \times \mathcal{SE}$ *and has no analytic section.*

Proof. First we notice that, for two separable Banach spaces X and Y, $X \cong Y$ if and only if there are $\underline{x} \in X^\omega$ and $\underline{y} \in Y^\omega$ such that $\underline{x} \sim \underline{x}$ and $\overline{span}\{\underline{x}\} = X$ and $\overline{span}\{\underline{y}\} = Y$.

By Lemma 6.9, the subset $\{(X, Y, \underline{x}, \underline{y}) \ : \ \overline{span}\{\underline{x}\} = X,\ \overline{span}\{\underline{y}\} = Y,\ \underline{x} \sim \underline{x}\}$ is Borel in $\mathcal{SE} \times \mathcal{SE} \times C(2^\omega) \times C(2^\omega)$.

Since the image of this set under the natural projection onto $\mathcal{SE} \times \mathcal{SE}$ is just $\{(X, Y) \ : \ X \cong Y\}$, we get that $\cong$ is analytic. The the class $\langle \mathcal{U} \rangle$ is analytic not Borel. □

The family of Separable Reflexive Banach spaces

The next Theorem is a generalization of a deep result of J. Bourgain (see [3]).

Theorem 6.11. *Let $\mathcal{A}$ be an analytic family of separable Banach spaces, stable under isomorphism, which contains all separable reflexive spaces. Then $\mathcal{A}$ contains a space which is universal for all separable Banach spaces.*

Proof. Let φ the map defined in Lemma 6.7 above. We already know that, if θ is well founded then, by Theorem 6.6, $\phi(\theta) = U_2(\theta)$ is reflexive. Therefore, $\varphi^{-1}(\mathcal{A})$ is analytic and contains $\mathcal{WF}$. Since $\mathcal{WF}$ is far away to be analytic, there exists $\theta_0 \in \varphi^{-1}(\mathcal{A})$ which is ill founded. Thus, by Theorem 6.6 once again, $\varphi(\theta_0) = U_2(\theta_0)$ is an element of $\mathcal{A}$ which is isomorphic to $\mathcal{U}$. □

Corollary 6.12. *The family of all separable reflexive Banach spaces is co-analytic and not Borel.*

Proof. It is enough notice that this family cannot contains $\mathcal{U}$ (actually $\mathcal{U}$ contains a complemented copy of ℓ_1). Thus, by Theorem 6.11, this family cannot be analytic, thus not Borel. □

Theorem 6.13. *(J. Bourgain, 1980)*
If X is separable and universal for the set of all separable reflexive Banach spaces, then X is also universal for the class of all separable Banach spaces.

Proof. Since if X is universal for the set of all separable reflexive Banach space then the family $\mathcal{A}$, of all spaces which are isomorphic to a subspace of X, has to be analytic. Therefore by Theorem 6.11 X has to be universal. □

Actually, Bourgain's theorem was made to show an improvement of another celebrate result previously obtained. In the *Scottish book*, Banach and Mazur ([19, Problem 49]) arise the question of the existence of a universal Banach space more fine than $C(2^\omega)$: reflexive.

This was solved in a elegant way in 1968 by W. Slenk, introducing a new tool in Geometry: still called the *Slenk's index* (see [17]).

Theorem 6.14. *(W. Slenk, 1968)*
If X is a Banach space universal for all separable reflexive Banach space, then X^*, the dual of X, is non separable.

Proof. Let us give the idea of the proof.

For a weak* compact F of X^* and $\varepsilon > 0$ let us define:

$$\begin{aligned} F'_\varepsilon = \{f \in X^* \ : & \text{ there exist } (x_m)_m \subseteq B_X \text{ and} (f_m)_m \subseteq F \text{ such that} \\ & f_m \stackrel{w^*}{\to} f, \\ & x_m \to 0, \\ & \limsup_m |f_m(x_m)| \geq \varepsilon\} \end{aligned}$$

We denote

$$\begin{aligned} & F_\varepsilon^0 = F \\ & F_\varepsilon^{\alpha+1} = (F_\varepsilon^\alpha)'_\varepsilon \\ & F_\varepsilon^\alpha = \bigcap_{\beta<\alpha} F_\varepsilon^\beta, \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Moreover, let us set

$$S_\varepsilon(F) = \sup\{\alpha < \omega_1 : \ F_\varepsilon^\alpha \neq \emptyset\}.$$

Let us denote by $S_\varepsilon(X) = S_\varepsilon(B_{X^*})$.

It can be shown that, if X^* is separable, there exists an ordinal number $\alpha < \omega_1$ such that

$$F_\varepsilon^\alpha \neq \emptyset \quad \text{and} \quad F_\varepsilon^{\alpha+1} = \emptyset.$$

The proof is articulated in three step.

In the first step is proved that if X^* is separable, then

$$S(X) = \sup_n S_{\frac{1}{n}}(X) < \omega_1.$$

$S(X)$ is called the *Slenk index* of X.

It is showed that $S(X)$ is monotone: if a Banach space Y is isomorphic to a closed subspace of X then

$$S(Y) \leq S(X).$$

The last step consist to show that for every countable ordinal α there exists a separable Banach reflexive space X_α such that

$$S(X_\alpha) \geq \alpha. \tag{6.2}$$

This would finish the proof. Let us see the last step:

For two Banach space X and Y, let us denote by $(X \times Y)_1$ (respectively $(X \times Y)_\infty$) the Cartesian product of Banach spaces X and Y with the norm

$$\|(x, y)\|_1 = \|x\| + \|y\|$$
$$\text{(respectively} \quad \|(x, y)\|_\infty = \max(\|x\|,\ \|y\|)).$$

If $(X_t)_{t\in T}$ is a family of Banach spaces, then the symbol $\ell_2(X_t)_{t\in T}$ denote the Banach space of all function $x(\cdot)$ from X into the product $\prod_{t\in T} X_t$ such that

$$\|x(\cdot)\|_2 = (\sum_{t\in T} \|x(t)\|^2)^{\frac{1}{2}} < \infty.$$

It is not hard to show that, if X^* is separable

$$S_\varepsilon((X \times \ell_2)_1) \geq S\varepsilon(X) + 1.$$

Now, let us set

$$\begin{aligned} &X_0 = \ell_2 \\ &X_{\alpha+1} = (X_\alpha \times \ell_2)_1 \\ &X_\alpha = \ell_2(X_\beta)_{\beta<\alpha}, \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Clearly, the spaces X_α defined above are reflexive and separable. By transfinite induction, we prove that the family $(X_\alpha)_{\alpha<\omega_1}$ satisfies (6.2).

Obviously, (6.2) holds for $\alpha = 0$. Suppose α is a limit ordinal and for $0 \leq \beta < \alpha$ (6.2) holds, using the fact that $X_\alpha = \ell_2(X_\beta)_{\beta<\alpha}$ contains a subspace isometrically isomorphic to X_β, by the second step we get

$$S_\varepsilon(X_\alpha) \geq \sup_{\beta<\alpha} S_\varepsilon(X_\beta) \geq \sup_{\beta<\alpha} \beta = \alpha.$$

Finally, if (6.2) holds for an ordinal $\alpha < \omega_1$ then

$$S_\varepsilon(X_{\alpha+1}) = S_\varepsilon((X_\alpha \times \ell_2)_1) \geq S_\varepsilon(X_\alpha) + 1 \geq \alpha + 1.$$

□

The family of Separable Uniformly Convex Banach spaces

A Banach space $(X, \|\cdot\|)$ is said to be *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in S_X$ with $\|x-y\| \geq \varepsilon$ we have that $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is a classical result that every uniformly convex Banach space is reflexive.

Using the Kuratowski-Rill Nardzewski Theorem 1.27, can be shown that if X is a separable Banach space then there exists a sequence of Borel maps

$$S_n : \mathcal{SE}(X) \longrightarrow X$$

such that

(a) if $Y = \{0\}$, then $S_n(Y) = 0$ for every $n \in \omega$;

(b) If $Y \in \mathcal{SE}(X)$ with $Y \neq \{0\}$, then $S_n(Y) \in S_Y$ for every $n \in \omega$;

(c) the sequence $S_n(Y))_n$ is norm dense in the sphere S_Y of Y.

Unlike the family of separable reflexive Banach spaces, we have

Theorem 6.15. *The family of separable uniformly convex Banach spaces is Borel.*

Proof. Let $S_n : \mathcal{SE} \longrightarrow C(2^\omega)$ as above.

It is enough to note that

$$X \in \mathcal{SE} \text{ is uniformly convex} \Leftrightarrow \forall n \in \omega \setminus \{0\} \exists m \in \omega \setminus \{0\} \text{ such that } [\forall k, l \in \omega \text{ we have } \|S_k(X) - S_l(X)\| \geq \frac{1}{n} \Rightarrow \|\frac{S_k(X) + S_l(X)}{2}\| \leq 1 - \frac{1}{m}]$$

□

The family of Separable Banach space isomorphic to ℓ_2

As before, let us denote by $\langle \ell_2 \rangle$ the class of all Banach space which are isomorphic to the space ℓ_2.

A separable Banach space X is of *type 2* if there is some $M > 0$ such that for any finite sequence $(x_j)_{j=0}^n$ of elements of X we have

$$\frac{1}{2^n} \sum_{\epsilon_j = \pm 1} \|\sum_{j=0}^n \epsilon_j x_j\| \leq M \, (\sum_{j=0}^n \|x_j\|^2)^{\frac{1}{2}}.$$

A separable Banach space X is of *cotype 2* if there is some $M > 0$ such that for any finite sequence $(x_j)_{j=0}^n$ of elements of X we have

$$\frac{1}{2^n}\sum_{\epsilon_j=\pm 1}\|\sum_{j=0}^n \epsilon_j x_j\| \geq \frac{1}{M}\ (\sum_{j=0}^n \|x_j\|^2)^{\frac{1}{2}}.$$

The following result is due to S. Kwapień

Theorem 6.16. *([10])*
A separable Banach space is isomorphic to ℓ_2 if and only if is of type 2 and cotype 2.

Now, for any $\underline{v} \in C(2^\omega)^\omega$, the space $\overline{span}(\underline{v})$ is of type 2 if and only if there is some $M \in \mathbb{Q}^+$ such that for any $n \in \omega$ and any $(\lambda^j)_{j=0}^n \in (\mathbb{Q}^{<\omega})^{n+1}$, we have

$$\frac{1}{2^n}\sum_{\epsilon_j=\pm 1}\|\sum_{j=0}^n \epsilon_j \lambda^j \cdot \underline{v}\| \leq M\ (\sum_{j=0}^n \|\lambda^j \cdot \underline{v}\|^2)^{\frac{1}{2}}.$$

where $\lambda^j \cdot \underline{v} = \sum_i \lambda_i^j v_i$. Consequently, $\{\underline{v} \in C(2^\omega)^\omega\ :\ \overline{span}(\underline{v})$ is of type $2\}$ is Borel. Similarly, can be proved that $\{\underline{v} \in C(2^\omega)^\omega\ :\ \overline{span}(\underline{v})$ is of cotype $2\}$ is Borel. Using Kwapień therem, we have the following

Theorem 6.17. *The class $\langle \ell_2 \rangle$ is Borel.*

It follows from Bourgain's work (see [4]) that the equivalent classes $\langle L_p(0,1)\rangle$ when $1 < p < \infty$ and $p \neq 2$ are not Borel.

Question 6.18. Is there some separable Banach space X such that X is not isomorphic to ℓ_2 and the equivalent class $\langle X \rangle$ is Borel?

The family of Separable Reflexive Banach spaces: another approach

Let $X \in \mathcal{SB}$, $\varepsilon > 0$ and $K \geq 1$. We define a tree $T = T(X, \varepsilon, K)$ on S_X the shere of X, given by

$$(x_n)_{n=0}^l \in T \iff (x_n)_{n=0}^l \text{ is } K\text{-Schauder and } \forall a_0, \ldots, a_l \in \mathbb{R}_+ \\ \text{with } \sum_{n=0}^l a_n = 1 \text{ we have } \|\sum_{n=0}^l a_n x_n\| \geq \varepsilon$$

where a finite sequence $(x_n)_{n=0}^l$ is said to be K-Schauder if

$$\|\sum_{n=0}^{l} a_n x_n\| \leq K \cdot \|\sum_{n=0}^{l} a_n x_n\|$$

for every $0 \leq m \leq l$ and every $a_0, \ldots, a_n \in \mathbb{R}$.

Notice that if $0 < \varepsilon' < \varepsilon$ and $1 \leq K < K'$, then the tree $T(X, \varepsilon, K)$ is a downwards closed subtree of $T(X, \varepsilon', K')$.

Theorem 6.19. *Let $X \in \mathcal{SB}$. Then X is reflexive if and only if for every $\varepsilon > 0$ and every $K \geq 1$ the tree $T(X, \varepsilon, K)$ is well founded.*

Proof. Let $\varepsilon > 0$ and $K \geq 1$, and assume, first, that the tree $T = T(X, \varepsilon, K)$ is not well founded.

Then, there exists a sequence (x_n) in X such that $(x_n)_{n=0}^l \in T$ for every $l \in \omega$. Notice that (x_n) is a normalized basic sequence. By Rosenthal 's Dichotomy [15], either there exists $L = \{l_0 < l_1 > \ldots\} \in [\omega]$ such that (x_{l_n}) is equivalent to the standart unit vector basis of ℓ_1, or there exist $M = \{m_0 < m_1 < \ldots\} \in [\omega]$ and $x^{**} \in X^{**}$ such that the sequence (x_{m_n}) is weak* convergent to x^{**}. In the first case, we imediately get that X is not reflexive. In the second case we distingush two subcases. If $x^{**} \in X^{**} \backslash X$, then clearly X is not reflexive. So assume thta $x^{**} \in X$. As (x_{m_n}) is basic, we see that $x^{**} = 0$ (i.e., the sequence is weakly null). By Mazur's Theorem, there exists a finite convex combination z of $\{x_{m_n} : n \in \omega\}$ such that $\|z\| < \varepsilon$. This is clearly impossible for the definition of the tree $T = T(X, \varepsilon, K)$. Then, X has to be not reflexive.

Conversely, assume that X is not reflexive. There exists $x^{**} \in X^{**} \setminus X$ with $\|x^{**}\| = 1$. If ℓ_1 embeds into X, then we can find ε and K such that $T(X, \varepsilon, K)$ is not well founded. If ℓ_1 doesn't embed into X, then by Odell-Rosenthal's theorem [13] there exists a sequence (z_n) in B_X which is weak* convergent to x^{**}. We may select $x^* \in X^*$ with $\|x^*\| \leq 1$ and $L \in [\omega]$ such that $x^*(z_n) \geq \frac{1}{2}$ for every $n \in L$. Notice that $\frac{1}{2} \leq \|z_n\| \leq 1$ for every $n \in L$. By a classical result of Bessaga-Pelczynski (see [5], p. 41) there exists $M \in [L]$ such that the sequence (z_{m_n}) is basic with basis constant $K \geq 1$. We set $x_n = \frac{z_{m_n}}{\|z_{m_n}\|}$ for every $n \in \omega$. Then (x_n) is a normalized basic sequence with basic constant K. Moreover, for every $l \in \omega$ and every $a_0, \ldots, a_l \in \mathbb{R}_+$ with $\sum_{n=0}^{l} a_n = 1$ we have

$$\|\sum_{n=0}^{l} a_n x_n\| \geq \sum_{n=0}^{l} a_n \frac{x^*(z_{m_n})}{\|z_{m_n}\|} \geq \frac{1}{2}$$

It follows that $(x_n)_{n=0}^l$ set in $T(X, \frac{1}{2}, K)$ for every $l \in \omega$. That means $T = T(X, \frac{1}{2}, K)$ is not well founded. □

Let $S_n : \mathcal{SB} \longrightarrow C(2^\omega)$ be the sequence of Borel map as in Theorem 6.15. We define the tree $T(X, j, k)$ on ω by the rule

$$(n_0, \dots, n_l) \in T(X, j, k) \Longleftrightarrow (S_{n_0}(X), \dots, S_{n_l}(X)) \in T(X, \frac{1}{j}, k).$$

We have the following

Lemma 6.20. *For every $j, k \geq 1$ the map $X \longmapsto T(X, j, k)$ is Borel.*

Proof. It is enough to show that for every $t = (n_0, \dots, n_l) \in \omega^{<\omega}$ the set

$$A_t = \{X \in \mathcal{SB} : \ t \in T(X, j, k)\}$$

is Borel.

We have

$$A \in A_t \Longleftrightarrow (S_{n_0}(X), \dots, S_{n_l}(X)) \text{ is } k\text{-Schauder and } \forall a_0, \dots, a_l \in \mathbb{Q}_+ \text{ with } \sum_{i=0}^{l} a_i = 1 \text{ we have } \|\sum_{i=0}^{l} a_i S_{n_i}(X)\| \geq \frac{1}{j}$$

As the sequence (S_n) consists of Borel maps, we conclude that the set A_t is Borel. □

With the family of trees $\{T(X, j, k) \ : \ j, k \geq 1\}$we produce a tree $T(X)$ on ω defined by the rule

$$p \frown t \in T(X) \Longleftrightarrow p = \langle j, k \rangle \text{ and } t \in T(X, j, k)$$

We can conclude with the following

Theorem 6.21. *The family of all separable reflexive Banach space is a Π^1_1 set in $\mathcal{F}(C(2^\omega))$ and the map*

$$X \longmapsto ht(T(X))$$

is a Π^1_1-rank map on $\mathcal{SBR}$.

The family of Separable Banach spaces with separable dual

This section is devote to the study of the set

$$\mathcal{SD} = \{X \in \mathcal{SB} \ : \ X^* \text{ is separable}\}.$$

Let us recall, once again, the notion of Slenk's index.

Let Z be a separable Banach space. Also let $\varepsilon > 0$ and K be a weak* cmpact subset of B_{Z^*}. We define

$$s_\varepsilon(K) = K \setminus \bigcup\{V \subseteq Z^* : \ V \text{ is weak}^* \text{ open and } \|\cdot\| - diam(K \cap V) \leq \varepsilon\}$$

where

$$\|\cdot\| - diam(A) = \sup\{\|z^* - y^*\| : z^*, y^* \in A\}$$

for every $A \subseteq Z^*$.

Notice that $s_\varepsilon(K)$ is weak* closed, $s_\varepsilon(K) \subseteq K$ and $s_\varepsilon(K_1) \subseteq s_\varepsilon(K_2)$ if $K_1 \subseteq K_2$. It follows that s_ε is a derivate on the set all weak* compact subsets of (B_{Z^*}, w^*). Hence, by transfinite recursion, for every weak* compact subset K of B_{Z^*} we define the iterate derivatives on K by

$$\begin{aligned} &s_\varepsilon^0(K) = K \\ &s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K)) \\ &s_\varepsilon^\lambda(K) = \bigcap_{\xi<\lambda} s_\varepsilon^\xi(K) \text{ if } \lambda \text{ is limit.} \end{aligned}$$

Let $S_{Z_\varepsilon}(K)$ the least ordinal ξ such that $s_\varepsilon^\xi(K) = \emptyset$, and $s_{Z_\varepsilon}(K) = \omega_1$ otherwise. The *Slenk's index* of Z is defined by

$$S_Z(Z) = \sup\{S_{Z_\varepsilon}(B_{Z^*}) \ : \ \varepsilon > 0\}.$$

It is easy to see that if $0 < \varepsilon_1 < \varepsilon_2$, then $S_{Z_{\varepsilon_1}}(K) \geq S_{Z_{\varepsilon_2}}(K)$, and so

$$S_Z(Z) = \sup\{S_{Z_{\frac{1}{n}}}(B_{Z^*}) \ : \ n \geq 1\}.$$

for every separable Banach space Z. Let us recall the main properties of the Slenk's index

Proposition 6.22. *Let Z and Y be two separable Banach spaces. Then the following hold:*

(i) If Y is isomorphic to Z, then $S_Z(Y) = S_Z(Z)$;

(ii) If Y is isomorphic to a closed subspace of Z, then $S_Z(Y) \leq S_Z(Z)$;

(iii) The dual Z^* of Z is separable if and only if $S_Z(Z) < \omega_1$.

Part (*i*) and (*ii*) are easy consequence of the definition. Part (*iii*) follows from

Lemma 6.23. *Let Z be a separable Banach space and K be a non-empty weak* compact subset of B_{Z^*}. If K is norm separable, then for every $\varepsilon > 0$ there exists a weak* open subset V of Z^* such that $K \cap V \neq \emptyset$ and*

$$\|\cdot\| - diam(K \cap V) \leq \varepsilon.$$

Proof. Let us consider a compatible metric ρ for (B_{Z^*}, w^*), with $\rho - diam(B_{Z^*}) \leq 1$. Suppose the assertion of the lemma is false. Then we may construct a family (V_t) $(t \in 2^{<\omega})$ of relatively weak* open subsets of K such that for every $t \in 2^{<\omega}$, setting F_t to be the weak* closure of V_t, the following hold

(a) $F_{t^\frown 0} \cap F_{t^\frown 1} = \emptyset$

(b) $F_{t^\frown 0} \cup F_{t^\frown 1} \subseteq V_t$

(c) $\rho - diam(V_t) \leq 2^{-|t|}$

(d) for every $n \geq 1$, every $t, s \in 2^n$ with $t \neq s$ and every pair $(z^*, y^*) \in V_t \times V_s$ we have $\|z^* - y^*\| > \varepsilon$.

We set $P = \bigcup_{\sigma \in 2^\omega} \bigcap_{n \in \omega} V_{\sigma|n}$.

By $(a), (b)$ and (c) we have that P is perfect subset of K. By (d) we get that $\|z^* - y^*\| > \varepsilon$ for every $z^*, y^* \in P$ with $z^* \neq y^*$. That implies that K is not norm-separable, a contradiction. □

Now, we want to show, first, that the Slenk's index is a Π^1_1-rank map on the set of all compact and norm separable subset of $E = (B_{Z^*}, w^*)$.

We fix $(V_m)_m$ a basis of open set in (B_{Z^*}, w^*). For every $n, m \in \omega$ define the map

$$D_{n,m} : \mathcal{K}(E) \longrightarrow \mathcal{K}(E)$$

by

$$D_{n,m}(K) = \begin{cases} K \setminus V_m, & \text{if } K \cap V_m \neq \emptyset \text{ and } \|\cdot\| - diam(K \cap V_m) \leq \frac{1}{n+1}; \\ K, & \text{otherwise.} \end{cases}$$

Notice that $D_{n,m}$ is a derivative on $\mathcal{K}(E)$. Now define

$$D_n : \mathcal{K}(E) \longrightarrow \mathcal{K}(E)$$

by

$$D_n(K) = \bigcap_m D_{n,m}(K)$$

. Observe that

$$D_n(K) = K \setminus \bigcup \{V \subseteq E : \ V \text{ is open and } \|\cdot\| - diam(K \cap V) \leq \frac{1}{n+1}\}$$

Clearly D_n is a derivative too.

Lemma 6.24. *For every $n \in \omega$, the map D_n is Borel.*

Proof. For every $m \in \omega$ let us consider

$$A_m = \{K \in \mathcal{K}(E) : \ K \cap V_m \neq \emptyset \text{ and } \|\cdot\| - diam(K \cap V_m) \leq \frac{1}{n+1}\}$$

As the norm of Z^* is lower semicontinuous, it follows that A_m is a Borel subset of $\mathcal{K}(E)$. Now observe that

$$D_{n,m}(K) = K \text{ if } K \notin A_m \text{ and}$$
$$D_{n,m}(K) = K \setminus V_m \text{ if } K \in A_m.$$

This easily implies that the map $D_{n,m}$ is Borel for every $m \in \omega$.

Now consider the map

$$F : (\mathcal{K}(E))^\omega \longrightarrow (\mathcal{K}(E))^\omega$$

defined by

$$F((K_m)) = (D_{n,m}(K_m)).$$

Then F is Borel. In particular the map $\bigcap : (\mathcal{K}(E))^\omega \longrightarrow \mathcal{K}(E)$ defined by $\bigcap((K_m)) = \bigcap_m K_m$ is Borel too. Finally, considering the continuous map $I : \mathcal{K}(E) \longrightarrow (\mathcal{K}(E))^\omega$ defined by $I(K) = (K_m)$ with $K_m = K$ for each $m \in \omega$, we conclude that D_n is Borel from the equality

$$D_n(K) = \bigcap(F(I(K)))$$

□

By Theorm4.4, we have that

$$\Omega_Z = \{K \in \mathcal{K}(E) : \ D_n^\infty(K) = \emptyset \ \forall n \in \omega\}$$

is Π_1^1, and the map

$$K \longmapsto \sup\{|K|_{D_n} : \ n \in \omega\}$$

is a Π_1^1-rank map.

Observe that, by the Lemma 6.23,

$$\Omega_Z = \{K \in \mathcal{K}(E) : \ K \text{ is norm-separable}\}$$

and

$$S_Z(K) = \sup\{|K|_{D_n} : \ n \in \omega\}$$

for K in Ω_Z.

Theorem 6.25. *The set $\mathcal{SD}$ is Π^1_1 and the map*

$$X \longmapsto S_Z(X)$$

is a Π^1_1-rank map.

Proof. First, let us consider $Z = \ell_1$ and $H = (B_{\ell_\infty}, w^*)$. From what we have said above

$$\Omega = \{K \in \mathcal{K}(H) : \ K \text{ is norm-separable}\}$$

is Π^1_1 and the map $k \longmapsto \sup\{|K|_{D_n} : \ n \in \omega\}$ is Π^1_1-rank map on Ω.

Let D be the Borel subset of $\mathcal{SB} \times H$ defined as

$$(X, f) \in D \Longleftrightarrow f \in K_X = \{f_{x^*} \ : \ x^* \in B_{X^*}\},$$

where

$$f_{x^*} = (\frac{x^*(d_0(X))}{\|d_0(X)\|}, \dots, \frac{x^*(d_n(X))}{\|d_n(X)\|}, \dots)$$

with (d_n) the Borel maps form the Kuratowski and Ryll- Nardzewski's theorem.

For every $X \in \mathcal{SB}$ the section $D_X = \{f : \ (X, f) \in D\}$ is compact and equals to the set K_X. By Theorem 4.5 the map

$$\Phi : \mathcal{SB} \longrightarrow \mathcal{K}(H)$$

defined by

$$\Phi(X) = K_X$$

is Borel. Notice that

$$X \in \mathcal{SD} \Longleftrightarrow \Phi(X) \in \Omega.$$

It follows that the set $\mathcal{SD}$ is Π^1_1 and the map

$$X \longmapsto \sup\{|K_X|_{D_n} : \ n \in \omega\}$$

is Π^1_1-rank on $\mathcal{SD}$.

To finish, it is enough to note that for every $X \in \mathcal{SD}$ and $n \in \omega$ we have $|K_X|_{D_n} = S_{Z_{\frac{1}{n}}}(B_{X^*})$. Hence

$$\sup\{|K_X|_{D_n} : \ n \in \omega\} = \sup\{S_{Z_{\frac{1}{n}}}(B_{X^*}) : \ n \geq 1\} = S_Z(X).$$

□

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